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AN EFFICIENT FORMULATION FOR THE MODELING OF GENERAL MULTI-FLEXIBLE-BODY CONSTRAINED SYSTEMS

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Abstract—This paper presents a computer based method for the efficient formulation and solution of the nonlinear equations of motion for mechanical systems which are modeled as systems of interconnected flexible bodies and are subject to motion and/or geometric constraints. Flexibility is modeled through the use of admissible functions obtained from previous finite element analysis of the component bodies. The procedure then forms the equations of motion and solves for the system state derivatives associated with the unconstrained system in a highly efficient, *order n* manner. The constraint loads required to enforce the constraint relations are subsequently determined through the use of a constraint stabilization method. The required constraint loads are then used to modify the state derivatives found previously, resulting in a set of state derivative values which are now associated with the constrained system. The procedure is efficient and produces simulation computer code which is highly parallel in form and lends itself well to application on parallel computers.

1. INTRODUCTION

Driven by the needs of the aerospace, robotics and machine design industries, the last three decades have seen great advances in areas of dynamic simulation and analysis. In the area of robotics, the modeling of high performance robotic manipulators as systems of rigid bodies is often insufficient. Joint compliance and structural flexibility of the component parts must also be considered for adequate dynamic simulation and analysis. In the area of spacecraft design, many proposed structures are large and limber. Due to their size, development and verification testing of these structures in the laboratory is, and will likely continue to be, impractical if not impossible. Even if such tests could be made, results obtained in the earth's gravity and air environment could well be misleading or inconclusive regarding the structure's in-orbit behavior. For these reasons, analytic modeling and simulation are essential tools in large space system design.

For analytical modeling and computer simulation to be effective tools, they must be both fast and economical. The design process may require many simulations, and may thus be limited by time and/or monetary constraints. Much work has been done in the development of simulation procedures which are sufficiently general to handle a wide variety of multibody systems. However, the computational cost associated with many of these methods is considerable, thus limiting the extent to which they may be applied. As a result emphasis must also be placed on developing algorithms and simulation programs which are computationally efficient/economical, while remaining general enough to adequately simulate a wide variety of systems. Presented in this paper is one such approach, based on a highly efficient *Order n*, $O(n)$, algorithm for dealing with multibody dynamic systems, which considers geometric stiffening (too often neglected in many formulations) and allows for the presence of closed loops with any or all of the bodies of the system being flexible.

Earlier models of multibody systems, using finite element or assumed mode methods, were based on the assumption that small deformations of the bodies do not affect the nominal rigid body motion significantly (Imam and Sandor, 1975; Bahat and Willmert, 1976; Sunada and Dubowsky, 1981). In their analysis the inertial and reaction forces were evaluated from rigid motion of the component bodies of the system and introduced to the linear elastic problem as external forces for computing the corresponding deflections. The elastic deformation however does not yield accurate results for situations in which the dynamic coupling of the rigid and flexible motion is significant.

Analysis procedures developed by Agrawal and Shabana (1985), as well as Yoo and Haug (1986), involve formulation of the equations of motion of each elastic body in terms of its *absolute* rigid body and flexible degrees of freedom. The rigid body motion and elastic deformations are then solved for simultaneously. However, the interactions of the bodies are described by a large set of constraint equations formulated for each type of joint. This procedure can increase the dimension of the problem considerably and the introduction of Lagrange multipliers associated with constraint forces from the equations of motion requires costly computations for updated transformations.

Singh *et al.* (1984) used a formulation based on "Kane's Method" as found in Kane and Levinson (1980). This approach incorporated flexibility through assumed mode shapes obtained from previous finite element analysis of each body. This approach had some recursive aspects, but yielded equations of motion which were highly coupled in the generalized coordinates. The approach was limited to clamped-free mode shapes and the analysis of open tree configurations. More recently, work done by Bae and Haug (1986a, 1987b), Kim and Haug (1988, 1989), Wehage (1988) and VanderVoort (1988) has placed much greater emphasis on computational speed, efficiency and the use of parallel processing.

However, with many of these procedures which achieve model reduction through the use of modal coordinates, the derivation underlying them contains terms which have been prematurely linearized in the modal coordinates. This can lead to grossly incorrect simulations as shown by Kane *et al.* (1987). When corrective terms are added to account for the premature linearization of expressions, correct dynamic behavior of the system is predicted (Banerjee and Dickens, 1990; Wallrapp and Schwertassek, 1991).

Amirouche and Ider (1989) presented an approach utilizing relative coordinates for rigid-body degrees of freedom and assumed mode shapes which is also based on Kane's formulation. The approach is $O(n^3)$ and is applicable to both tree and closed loop configurations. The paper discusses the need for producing geometric stiffening terms to correct for the premature linearization in the modal terms and presents a method for generating these corrective terms when dealing with beams.

An alternate approach is presented in this paper. The recursive method uses relative coordinates for rigid body displacements, and shape functions obtained from finite element analysis for the representation of elastic body flexibility. In addition, geometric stiffening corrective terms can be determined for all flexible bodies. The resulting procedure is efficient, can be applied to both tree and closed loop configurations, and lends itself well to application on loosely coupled distributed architecture parallel computers.

2. ANALYTICAL DEVELOPMENT

2.1. *Notation and geometry*

Throughout this paper, scalar quantities will be represented as italics. Vector quantities will be denoted by bold symbols, while matrices are shown as italics with an underbar. Dyadics are represented by bold faced symbols with an under tilde, while matrices composed of either vector or dyadic quantities are represented by bold faced symbols with an underbar. For example, the symbols A, $\mathbf{A}$, $\underline{A}$, $\utilde{\mathbf{A}}$ and $\underline{\mathbf{A}}$, represent a scalar, vector, matrix, dyadic and matrix of vector or dyadic quantities, respectively. Unless stated otherwise, a variable in the subscript preceded by a comma indicates differentiation with respect to that variable, and summations are carried out over repeated indices, with indices $l, m, o, p = 1, 2, 3$; $q = 1, \ldots, 21$; $r, s, t = 1, \ldots, \mu^{B^k}$; and $j = 1, \ldots, \mu^{J^k}$. For each of these, μ^{B^k} and μ^{J^k} are the number of flexible degrees of freedom associated with body B^k, and the number of rigid body degrees of freedom associated with the kth joint, J^k.

Consider a multibody system consisting of $\mathcal{N}$ rigid and flexible bodies as depicted by Fig. 1. The bodies of the system are interconnected by joints allowing from one to six degrees of freedom. The system may be a tree configuration or may contain one or more closed loops. If closed loops exist in the system, the bodies of the loop are numbered as though the system were in a tree configuration with the associated loop being cut at a joint selected by the analyst.

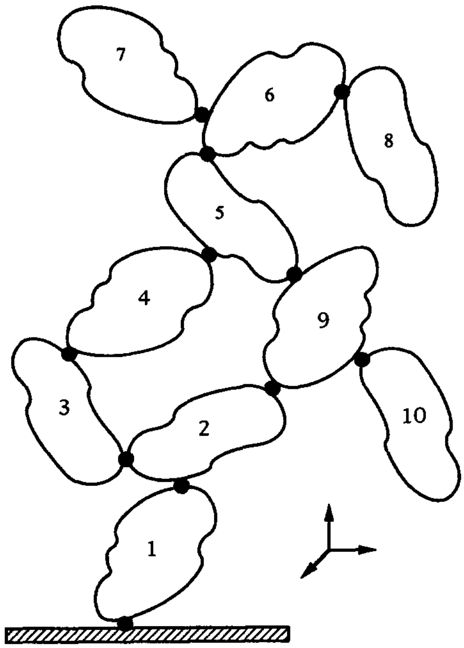


Fig. 1. Generic multibody system.

The algorithm to be presented assumes the following rules are obeyed in numbering the bodies: (i) every body has a higher number than its proximal body; (ii) consecutive integers are used in the numbering of bodies, but adjacent bodies need not have consecutive numbers.

Once the identification number for each body in the system has been properly assigned, the topology of the system is uniquely defined by identification numbers of the proximal bodies. The notation $Pr[k]$ refers to the set containing only the proximal body of the B^k. *Pr* is called the *proximal body array* and the convention that

$$Pr[k] = 0, \quad \text{if } k \text{ is a base body} \tag{1}$$

is used where bodies 0 and N are synonymous labels for the inertial reference frame.

It is useful to introduce additional sets to characterize other aspects of the system's topology. The notation $An[k]$ refers to a set of body identification numbers defined as

$$An[k] \triangleq \{j \mid \text{numbers of all bodies lying between body } k \text{ and body } 0\} \tag{2}$$

and is called the *ancestor body set array*.

The notation $Dist[k]$ refers to a set of body numbers defined as

$$Dist[k] \triangleq \{j \mid Pr[j] = k\}. \tag{3}$$

Dist is called the *distal body set array*. A body k for which $Dist[k] = \varnothing$ is called a *terminal body* of the system. The notation $Des[k]$ refers to a set of body identification numbers defined as

$$\begin{aligned} Des[k] &\triangleq \bigcup_{j \in Dist[k]} (\{j\} \cup Des[j]) \\ &= \{j \mid \text{all bodies which are outboard of body } k\}. \end{aligned} \tag{4}$$

Des is called the *descendent body set array*.

The description of system topology given by *Pr*, *An*, *Dist* and *Des* is redundant because, given any one of these arrays, one can construct the remaining three.

A typical flexible body of the system is defined as a body B^k which undergoes relative rigid body motion with respect to its proximal body $B^{Pr[k]}$ and deforms elastically (Fig. 2). Let $\mathbf{n}^{k,0}$ be the reference frame of B_k with respect to which the deformation of the body is given. Reference frame $\mathbf{n}^{k,i}$ represents the frame within which the dextral mutually perpendicular unit vectors $\mathbf{n}_1^{k,i}$, $\mathbf{n}_2^{k,i}$, $\mathbf{n}_3^{k,i}$ are fixed and which is itself fixed with respect to point $P^{k,i}$, the ith grid point of the finite element representation of B^k. The point for which $i = J^k$, is that point of B^k which connects B^k to $B^{Pr[k]}$ through joint J^k.

The position of an arbitrary $P^{k,i}$ of B^k with respect to P^{k,J^k} is given by the position vector

$$\boldsymbol{\xi}^{k,i} \triangleq \mathbf{r}^{k,i} + \hat{\boldsymbol{\phi}}_s^{k,i} q_s^{B^k} - (\mathbf{r}^{k,J^k} + \hat{\boldsymbol{\phi}}_s^{k,J^k} q_s^{B^k}), \tag{5}$$

where $\mathbf{r}^{k,i}$ is the position vector of $P^{k,i}$ in undeformed B^k and $\underline{\hat{\boldsymbol{\phi}}}^{k,i}$ is the admissible shape function matrix associated with the translation of $P^{k,i}$. The body B^k has μ^{B^k} modal deformation coordinates associated with it, where $q_s^{B^k}$ is the sth coordinate. Similarly, $q_s^{J^k}$ represents the sth of μ^{J^k} generalized coordinates associated with rigid body degrees of freedom arising from joint J^k.

For ease of notation, define the position vector from P^{k,J^k} to point $P^{k,i}$ in the undeformed state as $\boldsymbol{\rho}_{k,i}$. Similarly, let $\underline{\boldsymbol{\phi}}^{k,i}$ represent the modal deformation matrix at $P^{k,i}$ with respect to P^{k,J^k}.

Namely,

$$\boldsymbol{\rho}^{k,i} \triangleq \mathbf{r}^{k,i} - \mathbf{r}^{k,J^k} \tag{6}$$

and

$$\underline{\boldsymbol{\phi}}^{k,i} \triangleq \underline{\hat{\boldsymbol{\phi}}}^{k,i} - \underline{\hat{\boldsymbol{\phi}}}^{k,J^k}. \tag{7}$$

Similarly, the angular position of a particle at $P^{k,i}$ with respect to P^{J^k}, $\theta^{k,i}$, arising from the deformation of B^k, is expressed in terms of modal coordinates, $q_s^{B^k}$, and rotational

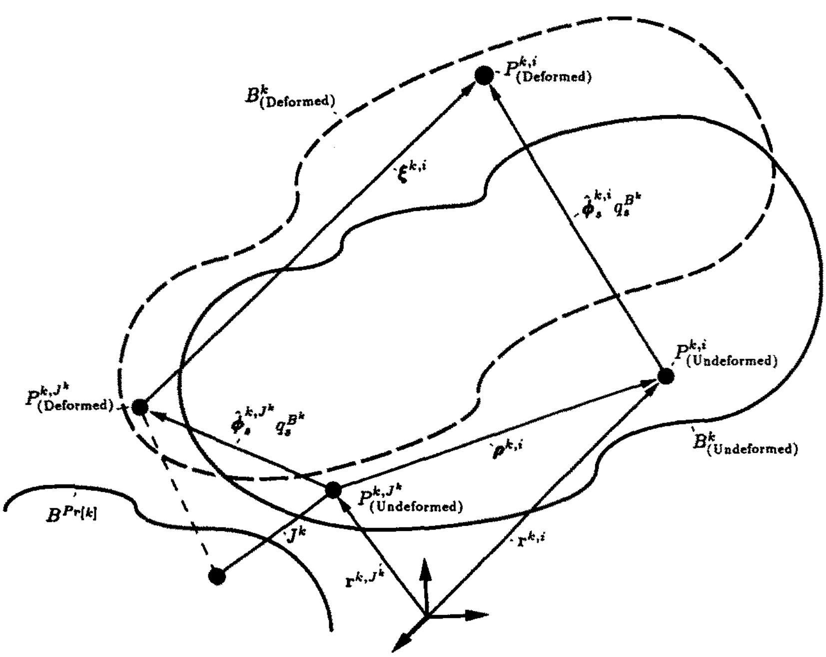


Fig. 2. Labeling convention for deformable bodies.

deformation shape functions. If $\psi_s^{k,i}$ is defined to be the rotational shape function associated with $q_s^{B^k}$ at $P^{k,i}$ with respect to P^{J^k}, then $\theta^{k,i}$ is given by

$$\theta_l^{k,i} \triangleq \psi_{ls}^{k,i} q_s^{B^k}. \tag{8}$$

Finally, unless otherwise specified, all time differentiations are taken in the inertia frame.

2.2. *Mathematical preliminaries*

To reduce the large number of elastic coordinates, a standard component mode technique as presented by Craig and Bampton (1968) will be utilized. This approach involves using a relatively small number of mode shapes for each elastic body. The mode shapes used for each body consist of selected free-vibration free-free modes, all constraint modes and any necessary static correction modes. The free vibration modes being extracted from the eigenvalue problem:

$$\mathcal{M}_{rs}^k \ddot{x}_s^k + \mathcal{K}_{rs}^k x_s^k = 0 \quad (r, s = 1, \ldots, \vartheta^k). \tag{9}$$

Where $\mathcal{M}^k$ and $\mathcal{K}^k$ are the structural mass and stiffness matrices obtained for B^k and ϑ^k is the total number of dynamic degrees of freedom associated with the full discretized flexible model of B^k. The approximate solution for eqn (9) is

$$x_s^k \approx \phi_{st}^k q_t^{B^k} \quad (s = 1, \ldots, \vartheta^k; \quad t = 1, \ldots, \mu^{B^k}), \tag{10}$$

where ϕ^k is the matrix of eigenvectors (mode shapes) associated with B^k, $q_t^{B^k}$ are the modal coordinates, and μ^{B^k} is the number of eigenvectors retained.

The free vibration modes selected (retained) are those modes which are anticipated to have the most significant modal contribution. The static corrections modes are shape functions associated with the application of a unit load at points of the body (with a clamped interface) where an external load, such as an actuator force, will be applied.

Partial velocities. Central to this approach is the use of partial angular velocities and partial velocities (Kane and Levinson, 1985). If we define $\boldsymbol{\omega}^P$ and $\mathbf{v}^P$ to be, respectively, the angular velocity and velocity of an arbitrary particle P of the system with respect to reference frame N, then $\boldsymbol{\omega}^P$ and $\mathbf{v}^P$ can be expressed as

$$\boldsymbol{\omega}^P = \sum_{r=1}^{n} \boldsymbol{\omega}_r^P u_r + \boldsymbol{\omega}_t^P \tag{11}$$

and

$$\mathbf{v}^P = \sum_{r=1}^{n} \mathbf{v}_r^P u_r + \mathbf{v}_t^P, \tag{12}$$

where, by definition,

$u_1, \ldots, u_n$ are *generalized speeds*, quantities which characterize the motion of the system,
$\boldsymbol{\omega}_r^P$ is the rth partial angular velocity of P in N,
$\mathbf{v}_r^P$ is the rth partial velocity of P in N.

Differentiating eqns (11) and (12) with respect to time yields

$$\boldsymbol{\alpha}^P = \frac{{}^N\mathrm{d}\boldsymbol{\omega}^P}{\mathrm{d}t} = \sum_{r=1}^{n} \boldsymbol{\omega}_r^P \dot{u}_r + \left(\sum_{r=1}^{n} \dot{\boldsymbol{\omega}}_r^P u_r + \dot{\boldsymbol{\omega}}_t^P \right) \tag{13}$$

and

$$\mathbf{a}^P = \frac{{}^N\mathrm{d}\mathbf{v}^P}{\mathrm{d}t} = \sum_{r=1}^{n} \mathbf{v}_r^P \dot{u}_r + \left(\sum_{r=1}^{n} \dot{\mathbf{v}}_r^P u_r + \dot{\mathbf{v}}_t^P \right) \tag{14}$$

for the angular acceleration of P in N, and the acceleration of P in N, respectively.

The quantities $\bar{\boldsymbol{\alpha}}^P$, $\hat{\boldsymbol{\alpha}}^P$, $\bar{\mathbf{a}}^P$ and $\hat{\mathbf{a}}^P$ are then defined to be

$$\bar{\boldsymbol{\alpha}}^P \triangleq \sum_{r=1}^{n} \boldsymbol{\omega}_r^P \dot{u}_r, \tag{15}$$

$$\hat{\boldsymbol{\alpha}}^P \triangleq \sum_{r=1}^{n} \dot{\boldsymbol{\omega}}_r^P u_r + \dot{\boldsymbol{\omega}}_t^P, \tag{16}$$

$$\bar{\mathbf{a}}^P \triangleq \sum_{r=1}^{n} \mathbf{v}_r^P \dot{u}_r, \tag{17}$$

$$\hat{\mathbf{a}}^P \triangleq \sum_{r=1}^{n} \dot{\mathbf{v}}_r^P u_r + \dot{\mathbf{v}}_t^P, \tag{18}$$

which will aid in the development and representation of recursive relationships discussed in the next section.

2.3. *Unconstrained systems*

Through the extensive use of recursive relationships, the state derivative values can be determined for general tree system in approximately $O(n)$ operations overall. The algorithm consists of three primary computational steps: (i) working from the base body outward toward the terminal bodies of the system, recursively determine kinematical quantities, specifically the angular velocities, velocities and acceleration remainder terms; (ii) working recursively inward from the terminal bodies to the base body, generate generalized active forces, the remainder term contribution to the generalized inertia forces, composite inertia values and triangularization of the resulting equations; (iii) recursively back substitute to generate state derivatives, $\dot{u}_1, \ldots, \dot{u}_n$.

(i) *Determination of kinematical quantities.* Some choices of generalized coordinates lend themselves much better to the production of recursive relationships than do others. The set of generalized coordinates which has shown itself to be best for the generation of recursive relationships consists of those coordinates which describe the relative orientation/position of adjacent bodies (Jain, 1989).

The mathematical model is constructed such that the joints between adjacent bodies are comprised of a series of properly oriented single degree of freedom revolute or prismatic "subjoints" connected via massless/dimensionless links. Thus, if joint J^k is a six degree of freedom *free joint*, then it would be described by the set of single degree of freedom subjoints $J_1^k, \ldots, J_6^k$. The value of each generalized coordinate associated with rigid body motion describes the placement of the subjoint/body with respect to its proximal subjoint/body. When joint J_j^k is revolute, the generalized coordinate $q_j^{J^k}$ represents the angle between sub-bodies J_j^k and J_{j-1}^k, with a joint axis parallel to unit vector $\boldsymbol{\lambda}^{J_j^k}$. If joint J_j^k is prismatic, then $q_j^{J^k}$ is the translation of sub-body J_j^k measured relative to subjoint J_{j-1}^k in the direction of unit vector $\boldsymbol{\lambda}^{J_j^k}$.

The generalized speeds, $u_1, \ldots, u_n$ used in this formulation are defined as

$$u_s^{B^k} \triangleq \dot{q}_s^{B^k} \quad \text{and} \quad u_j^{J^k} \triangleq \dot{q}_j^{J^k}. \tag{19}$$

With generalized coordinates and generalized speeds so defined, the following recursive kinematical relationships for angular velocity, velocity, partial angular velocity and partial velocity hold true:

$$\omega^{B^k,i} = \omega^{B^k,J^k} + \psi_s^{k,i} u_s^{B^k}, \tag{20}$$

$$\omega^{J_j^k} = \omega^{J_{j-1}^k} + \begin{cases} u_j^{J^k} \lambda^{J_j^k} & \text{if} \quad J_j^k \text{ is revolute,} \\ \mathbf{0} & \text{if} \quad J_j^k \text{ is prismatic.} \end{cases} \tag{21}$$

Similarly,

$$\mathbf{v}^{B^k,i} = \mathbf{v}^{B^k,J^k} + \omega^{B^k,J^k} \times \xi^{k,i} + \phi_s^{k,i} \mathbf{u}_s^{B^k}, \tag{22}$$

$$\mathbf{v}^{J_j^k} = \mathbf{v}^{J_{j-1}^k} + \begin{cases} \omega^{J_j^k} \times \gamma^{J_j^k} + u_j^{J^k} \lambda^{J_j^k} \times \gamma^{J_j^k} & \text{if} \quad J_j^k \text{ is revolute,} \\ u^{J_j^k} \lambda^{J_j^k} & \text{if} \quad J_j^k \text{ is prismatic,} \end{cases} \tag{23}$$

with γ_l^k representing the position vector from J_{l-1}^k to J_l^k.

For $\bar{\alpha}$ and $\bar{\mathbf{a}}$ we have

$$\bar{\alpha}^{B^k,i} = \bar{\alpha}^{B^k,J^k} + \psi_s^{k,i} \dot{u}_s^{B^k}, \tag{24}$$

$$\bar{\alpha}^{J_j^k} = \bar{\alpha}^{J_{j-1}^k} + \lambda^{J_j^k} \dot{u}_j^{J^k} \tag{25}$$

and

$$\bar{\mathbf{a}}^{B^k,i} = \bar{\mathbf{a}}^{B^k,J^k} + \bar{\alpha}^{B^k,J^k} \times \xi^{k,i} + \phi_s^{B^k,i} \dot{u}_s^{B^k}, \tag{26}$$

$$\bar{\mathbf{a}}^{J_j^k} = \bar{\mathbf{a}}^{J_{j-1}^k} + \bar{\alpha}^{J_{j-1}^k} \times \gamma^{J_{j-1}^k} + \lambda^{J_j^k} \dot{u}_j^{J^k}. \tag{27}$$

Similarly, for the angular acceleration remainder terms

$$\hat{\alpha}^{B^k,i} = \hat{\alpha}^{B^k,J^k} + \omega^{B^k,J^k} \times \psi_s^{ki,} u_s^{B^k}, \tag{28}$$

$$\hat{\alpha}^{J_j^k} = \hat{\alpha}^{J_{j-1}^k} + \begin{cases} \mathbf{u}_j^{J^k} (\omega^{J_j^k} \times \lambda^{J_j^k}) & \text{if} \quad J_j^k \text{ is revolute,} \\ \mathbf{0} & \text{if} \quad J_j^k \text{ is prismatic,} \end{cases} \tag{29}$$

and for $\hat{\mathbf{a}}$

$$\hat{\mathbf{a}}^{B^k,i} = \hat{\mathbf{a}}^{B^k,J^k} + \hat{\alpha}^{B^k,J^k} \times \xi^{k,i} + \omega^{B^k,J^k} \times (\omega^{B^k,J^k} \times \xi^{k,i}) + 2\omega^{B^k,J^k} \times \phi_s^{B^k,i} u_s^{B^k}, \tag{30}$$

$$\hat{\mathbf{a}}^{J_j^k} = \hat{\mathbf{a}}^{J_{j-1}^k} + \hat{\alpha}^{J_{j-1}^k} \times \gamma^{J_j^k} + \omega^{J_{j-1}^k} \times (\omega^{J_{j-1}^k} \times \gamma^{J_{j-1}^k}) + \begin{cases} \hat{\alpha}^{J_j^k} \times \gamma^{J_j^k} + (u_j^{J^k})^2 [\lambda^{J_j^k} \times (\lambda^{J_j^k} \times \gamma^{J_j^k})] & \text{if} \quad J_j^k \text{ is revolute,} \\ 2\omega^{J_j^k} \times u_j^{J^k} \lambda^{J_j^k} & \text{if} \quad J_j^k \text{ is prismatic.} \end{cases} \tag{31}$$

In all relations, the superscript "J_0^k" and "J_μ^k", are synonymous with the superscripts "$B^{Pr[k],J^k}$" and "$,J^k$", respectively.

Geometric stiffening. Linear strain energy theory assumes that the deformation components are independent. But in systems involving *high* rotation rates, high radial forces can occur and the coupling between radial and transverse deflections becomes significant. Unfortunately, the use of modal coordinates in modeling flexibility often results in equations of motion which are incorrectly linearized with respect to these coordinates. For this reason higher order strain energy terms need to be considered. These terms are used in the generation of geometric stiffness expressions which are added to the existing equations to correct for the premature linearization in the modal terms. The procedure used in this paper for the determination of geometric stiffening terms is that of Banerjee and Dickens (1990).

However, additions and modifications have been made to allow for multiple interconnected flexible bodies, and to make this procedure compatible with the $O(n)$ formulation.

As given by Banerjee and Dickens (1990), the element geometric stiffness is

$$\underline{k}_G^e = \int_e [\underline{N}_{,x}^T \underline{N}_{,y}^T \underline{N}_{,z}^T] \begin{bmatrix} \sigma_{xx0}\underline{U} & \sigma_{xy0}\underline{U} & \sigma_{xz0}\underline{U} \\ \sigma_{xy0}\underline{U} & \sigma_{yy0}\underline{U} & \sigma_{yz0}\underline{U} \\ \sigma_{xz0}\underline{U} & \sigma_{yz0}\underline{U} & \sigma_{zz0}\underline{U} \end{bmatrix} \begin{bmatrix} \underline{N}_{,x} \\ \underline{N}_{,y} \\ \underline{N}_{,z} \end{bmatrix} \mathrm{d}V, \tag{32}$$

where $\underline{N}(x, y, z)$ is the ($3 \times$ *e-ndof*) matrix of interpolation functions, with x, y, z as local coordinates. The stresses σ_{ij0} $(i, j = x, y, z)$ are those arising from the loads applied to the nodes of the element, the quantity *e-ndof* represents the number of element degrees-of-freedom, and $\underline{U}$ is a (3×3) identity matrix.

As discussed in Banerjee and Dickens (1990), finite element codes, such as NASTRAN, can compute the geometric stiffness associated with a prescribed distributed load by first calculating the associated element stresses. These element stresses are then used in the determination of the element geometric stiffness matrices as per eqn (32). Finally, the element geometric stiffness matrices are assembled into the body geometric stiffness matrix $\underline{K}_G^{B^k}$. Much of the necessary NASTRAN DMAP is available in the buckling analysis rigid formats and can be applied here with relatively little modification.

The overall geometric stiffness matrix for B^k, $\underline{K}_G^{B^k}$, is constructed from 21 time invariant contributing matrices, $\underline{\hat{K}}_{G_q}^{B^k}$ $(q = 1, \ldots, 21)$, and their associated temporal scalars, $A_{G_q}^{B^k}$, which arise from the inertia loads in B^k. In addition, there are another six contributing time invariant matrices, $\underline{\breve{K}}_{G_{jh}}^{B^k}$ $(j \in Dist[k]$; $h = 1, \ldots, 6)$, and their associated temporal scalars, $F_h^{J_1^j}$, which are affiliated with each distal body to B^k, and account for the geometric stiffening in B^k due to loads applied to it by each of these distal bodies. Specifically,

$$\underline{K}_G^{B^k} = \underline{\hat{K}}_{G_q}^{B^k}(\hat{A}_{G_q}^{B^k} + \bar{A}_{G_q}^{B^k}) + \sum_{j \in Dist[k]} \underline{\breve{K}}_{G_{jh}}^{B^k} F_h^{J_1^j} \quad (q = 1, \ldots, 21;\ j \in Dist[k];\ h = 1, \ldots, 6), \tag{33}$$

where $A_{G_q}^{B^k}$, and $F^{J_1^j}$, with their associated prescribed grid point loads for the generation of $\underline{\hat{K}}_{G_q}^{B^k}$ and $\underline{\breve{K}}_{G_{jh}}^{B^k}$, respectively, given in Table 1.

(ii) *Determination of generalized forces and triangularization of equations.* In general, for systems of $\mathcal{N}$ interconnected bodies, the equations of motion are given by Kane *et al.* (1983), as

$$\sum_{k=1}^{\mathcal{N}} \int_{V^k} \mathbf{v}_r^k \cdot \mathbf{R}^k \, \mathrm{d}V^k + \sum_{k=1}^{\mathcal{N}} \int_{V^k} \mathbf{v}_r^k \cdot \mathbf{R}^{k*} \, \mathrm{d}V^k = 0 \quad (r = 1, \ldots, \mathcal{N}). \tag{34}$$

The integration over the volume of each body and the indicated summations over all bodies of the system at each time step can be computationally expensive and the resulting equations of motion are highly coupled in the state derivatives. Consequently, the equations of motion must be triangularized and the state derivatives solved for, before the equations of motion can be temporally integrated.

Key to the reduction in the number of required operations is the removal of summations which appear above. The indicated summations over all bodies of the system can be reduced significantly through the use of relative coordinates for rigid body motion. Making full use of the recursive relations presented previously, the equation of motion can be rewritten as

$$\sum_{k \in Des[l] \cup \{l\}} \int_{V^k} \mathbf{v}_r^k \cdot \mathbf{R}^k \, \mathrm{d}V^k + \sum_{k \in Des[l] \cup \{l\}} \int_{V^k} \mathbf{v}_r^k \cdot \mathbf{R}^{k*} \, \mathrm{d}V^k = 0 \quad (r = 1, \ldots, N), \tag{35}$$

where l is the body or joint to which degree of freedom r is associated.

Table 1. Loadings for the generation of the geometric stiffness matrix

Temporal scalars	Load applied to grid i
Acceleration terms	Applied force
$\bar{A}^{B^k}_{G_1} = \bar{a}^{B^k,J^k}_1,\ \hat{A}_{G_1} = \hat{a}^{B^k,J^k}_1$	$m^{B^k,i}\mathbf{n}^{B^k,J^k}_1$
$\bar{A}^{B^k}_{G_2} = \bar{a}^{B^k,J^k}_2,\ \hat{A}_{G_2} = \hat{a}^{B^k,J^k}_2$	$m^{B^k,i}\mathbf{n}^{B^k,J^k}_2$
$\bar{A}^{B^k}_{G_3} = \bar{a}^{B^k,J^k}_3,\ \hat{A}_{G_3} = \hat{a}^{B^k,J^k}_3$	$m^{B^k,i}\mathbf{n}^{B^k,J^k}_3$
$\bar{A}^{B^k}_{G_4} = \bar{\alpha}^{B^k,J^k}_1,\ \hat{A}^{B^k}_{G_4} = \hat{\alpha}^{B^k,J^k}_1$	$(\rho^{B^k,i}_2\mathbf{n}^{B^k,J^k}_3 - \rho^{B^k,i}_3\mathbf{n}^{B^k,J^k}_2)m^{B^k,i}$
$\bar{A}^{B^k}_{G_5} = \bar{\alpha}^{B^k,J^k}_2,\ \hat{A}^{B^k}_{G_5} = \hat{\alpha}^{B^k,J^k}_2$	$(\rho^{B^k,i}_3\mathbf{n}^{B^k,J^k}_1 - \rho^{B^k,i}_1\mathbf{n}^{B^k,J^k}_3)m^{B^k,i}$
$\bar{A}^{B^k}_{G_6} = \bar{\alpha}^{B^k,J^k}_3,\ \hat{A}^{B^k}_{G_6} = \hat{a}^{B^k,J^k}_3$	$(\rho^{B^k,i}_1\mathbf{n}^{B^k,J^k}_2 - \rho^{B^k,i}_2\mathbf{n}^{B^k,J^k}_1)m^{B^k,i}$
$\hat{A}^{B^k}_{G_7} = (\omega^{B^k,J^k}_1)^2$	$(-\rho^{B^k,i}_2\mathbf{n}^{B^k,J^k}_2 - \rho^{B^k,i}_3\mathbf{n}^{B^k,J^k}_3)m^{B^k,i}$
$\hat{A}^{B^k}_{G_8} = \omega^{B^k,J^k}_1\omega^{B^k,J^k}_2$	$(\rho^{B^k,i}_2\mathbf{n}^{B^k,J^k}_1 + \rho^{B^k,i}_1\mathbf{n}^{B^k,J^k}_2)m^{B^k,i}$
$\hat{A}^{B^k}_{G_9} = \omega^{B^k,J^k}_1\omega^{B^k,J^k}_3$	$(\rho^{B^k,i}_3\mathbf{n}^{B^k,J^k}_1 + \rho^{B^k,i}_1\mathbf{n}^{B^k,J^k}_3)m^{B^k,i}$
$\hat{A}^{B^k}_{G_{10}} = (\omega^{B^k,J^k}_2)^2$	$(-\rho^{B^k,i}_1\mathbf{n}^{B^k,J^k}_1 - \rho^{B^k,i}_3\mathbf{n}^{B^k,J^k}_3)m^{B^k,i}$
$\hat{A}^{B^k}_{G_{11}} = \omega^{B^k,J^k}_2\omega^{B^k,J^k}_3$	$(\rho^{B^k,i}_2\mathbf{n}^{B^k,J^k}_3 + \rho^{B^k,i}_3\mathbf{n}^{B^k,J^k}_2)m^{B^k,i}$
$\hat{A}^{B^k}_{G_{12}} = (\omega^{B^k,J^k}_3)^2$	$(-\rho^{B^k,i}_1\mathbf{n}^{B^k,J^k}_1 - \rho^{B^k,i}_2\mathbf{n}^{B^k,J^k}_2)m^{B^k,i}$
$F^{J^j_1}_{l+3}$	$\mathbf{n}^{B^k,J^k}_l\ (l = 1, 2, 3)$
Angular acceleration terms	Applied moment
$\bar{A}^{B^k}_{G_{13}} = \bar{\alpha}^{B^k,J^k}_1,\ \hat{A}^{B^k}_{G_{13}} = \hat{\alpha}^{B^k,J^k}_1$	$I^{B^k,i}_{11}\mathbf{n}^{B^k,J^k}_1 + I^{B^k,i}_{12}\mathbf{n}^{B^k,J^k}_2 + I^{B^k,i}_{23}\mathbf{n}^{B^k,J^k}_3$
$\bar{A}^{B^k}_{G_{14}} = \bar{\alpha}^{B^k,J^k}_2,\ \hat{A}^{B^k}_{G_{14}} = \hat{\alpha}^{B^k,J^k}_2$	$I^{B^k,i}_{21}\mathbf{n}^{B^k,J^k}_1 + I^{B^k,i}_{22}\mathbf{n}^{B^k,J^k}_2 + I^{B^k,i}_{23}\mathbf{n}^{B^k,J^k}_3$
$\bar{A}^{B^k}_{G_{15}} = \bar{\alpha}^{B^k,J^k}_3,\ \hat{A}^{B^k}_{G_{15}} = \hat{\alpha}^{B^k,J^k}_3$	$I^{B^k,i}_{31}\mathbf{n}^{B^k,J^k}_1 + I^{B^k,i}_{32}\mathbf{n}^{B^k,J^k}_2 + I^{B^k,i}_{33}\mathbf{n}^{B^k,J^k}_3$
$\hat{A}^{B^k}_{G_{16}} = (\omega^{B^k,J^k}_1)^2$	$-I^{B^k,i}_{13}\mathbf{n}^{B^k,J^k}_2 + I^{B^k,i}_{12}\mathbf{n}^{B^k,J^k}_3$
$\hat{A}^{B^k}_{G_{17}} = \omega^{B^k,J^k}_1\omega^{B^k,J^k}_2$	$I^{B^k,i}_{13}\mathbf{n}^{B^k,J^k}_1 - I^{B^k,i}_{23}\mathbf{n}^{B^k,J^k}_2 + (I^{B^k,i}_{22} - I^{B^k,i}_{11})\mathbf{n}^{B^k,J^k}_3$
$\hat{A}^{B^k}_{G_{18}} = \omega^{B^k,J^k}_1\omega^{B^k,J^k}_3$	$-I^{B^k,i}_{12}\mathbf{n}^{B^k,J^k}_1 + (I^{B^k,i}_{11} - I^{B^k,i}_{33})\mathbf{n}^{B^k,J^k}_2 - I^{B^k,i}_{23}\mathbf{n}^{B^k,J^k}_3$
$\hat{A}^{B^k}_{G_{19}} = (\omega^{B^k,J^k}_2)^2$	$I^{B^k,i}_{23}\mathbf{n}^{B^k,J^k}_1 - I^{B^k,i}_{12}\mathbf{n}^{B^k,J^k}_3$
$\hat{A}^{B^k}_{G_{20}} = \omega^{B^k,J^k}_2\omega^{B^k,J^k}_3$	$(I^{B^k,i}_{33} - I^{B^k,i}_{22})\mathbf{n}^{B^k,J^k}_1 - I^{B^k,i}_{12}\mathbf{n}^{B^k,J^k}_2 - I^{B^k,i}_{13}\mathbf{n}^{B^k,J^k}_3$
$\hat{A}^{B^k}_{G_{21}} = (\omega^{B^k,J^k}_3)^2$	$-I^{B^k,i}_{23}\mathbf{n}^{B^k,J^k}_1 + I^{B^k,i}_{13}\mathbf{n}^{B^k,J^k}_2$
$F^{J^j_1}_{1}$	$\mathbf{n}^{B^k,J^k}_l\ (l = 1, 2, 3)$

$$A^{B^k,J^k}_j = \bar{A}^{B^k,J^k}_j + \hat{A}^{B^k,J^k}_j \quad (j = 1, \ldots, 21)$$

$$\bar{a}^{B^k,J^k}_l = \bar{\mathbf{a}}^{B^k,J^k}\cdot\mathbf{n}^{B^k,J^k}_l \quad \hat{a}^{B^k,J^k}_l = \hat{\mathbf{a}}^{B^k,J^k}\cdot\mathbf{n}^{B^k,J^k}_l \quad (l = 1, 2, 3)$$

$$\bar{\alpha}^{B^k,J^k}_l = \bar{\boldsymbol{\alpha}}^{B^k,J^k}\cdot\mathbf{n}^{B^k,J^k}_l \quad \hat{\alpha}^{B^k,J^k}_l = \hat{\boldsymbol{\alpha}}^{B^k,J^k}\cdot\mathbf{n}^{B^k,J^k}_l \quad (l = 1, 2, 3)$$

This procedure is still computationally expensive in formulating the equations of motion for all but the terminal bodies of the system and does not avoid the expense of solution for the state derivatives.

The indicated summation can be eliminated entirely through the recursive shifting of active and inertia forces to their proximal bodies. The state derivatives can then be determined inexpensively through the recursive triangularization of the resulting equations (Rosenthal, 1990; Anderson, 1990, 1991). The arguments of the integrals may be split into spatial and temporal parts, with the indicated spatial integrations needing to be carried out only once. The temporal quantities are then multiplied by time invariant coefficients resulting from the spatial integration.

If $\mathbf{n}^P_1$, $\mathbf{n}^P_2$ and $\mathbf{n}^P_3$ form a dextral set of mutually perpendicular unit vectors fixed with respect to P, then the mass and central inertia dyadics of P, $\underset{\sim}{\mathbf{M}}^P$ and $\underset{\sim}{\mathbf{I}}^P$, are defined as

$$\underset{\sim}{\mathbf{M}}^P \triangleq M^P\mathbf{n}^P_l\mathbf{n}^P_l, \tag{36}$$

where M^P is the mass lumped at P, and

$$\underset{\sim}{\mathbf{I}}^P \triangleq I^P_{lm}\mathbf{n}^P_l\mathbf{n}^P_m. \tag{37}$$

If $\boldsymbol{\rho}^P$ is written as

$$\boldsymbol{\rho}^P = \rho^P_1\mathbf{n}^P_1+\rho^P_2\mathbf{n}^P_2+\rho^P_3\mathbf{n}^P_3 \tag{38}$$

then we define the dyadic $\underset{\sim}{\boldsymbol{\rho}^P\times}$ as

$$\underset{\sim}{\boldsymbol{\rho}^P\times} \triangleq -\rho^P_3\mathbf{n}^P_1\mathbf{n}^P_2+\rho^P_2\mathbf{n}^P_1\mathbf{n}^P_3+\rho^P_3\mathbf{n}^P_2\mathbf{n}^P_1-\rho^P_1\mathbf{n}^P_2\mathbf{n}^P_3-\rho^P_2\mathbf{n}^P_3\mathbf{n}^P_1+\rho^P_1\mathbf{n}^P_3\mathbf{n}^P_2 \tag{39}$$

and let $\underset{\sim}{\mathbf{U}}$ be the *unit dyadic* given by

$$\underset{\sim}{\mathbf{U}} \triangleq \mathbf{n}^P_1\mathbf{n}^P_1+\mathbf{n}^P_2\mathbf{n}^P_2+\mathbf{n}^P_3\mathbf{n}^P_3. \tag{40}$$

For notational ease $\underline{\mathscr{P}}^{k,i}_r$, $\underline{\boldsymbol{\Phi}}^{J^k_j}_r$, $\underline{\mathscr{I}}_1$, $\underline{\mathscr{F}}_1$, $\underline{\mathscr{A}}$ and $\underline{\mathscr{S}}$ are introduced as

$$\underline{\mathscr{P}}^{J^k_j}_r \triangleq \begin{bmatrix}\boldsymbol{\omega}^{J^k_j}_r\\ \mathbf{v}^{J^k_j}_r\end{bmatrix},\quad \underline{\boldsymbol{\Phi}}^{k,i}_r \triangleq \begin{bmatrix}\boldsymbol{\psi}^{k,i}_r\\ \boldsymbol{\phi}^{k,i}_r\end{bmatrix}\quad\text{and}\quad \underline{\boldsymbol{\Phi}}^{k,i} \triangleq [\underline{\boldsymbol{\Phi}}^{k,i}_1,\ldots,\underline{\boldsymbol{\Phi}}^{k,i}_{\mu^{B^k}}], \tag{41}$$

$$\underline{\mathscr{I}}^{B^{k,i}}_1 \triangleq \begin{bmatrix}\underset{\sim}{\mathbf{I}}^{B^{k,i}} & \underset{\sim}{\mathbf{0}}\\ \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{M}}^{B^{k,i}}\end{bmatrix}\quad\text{and}\quad \underline{\mathscr{I}}^{J^k_j}_1 \triangleq \begin{bmatrix}\underset{\sim}{\mathbf{I}}^{J^k_j} & \underset{\sim}{\mathbf{0}}\\ \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{M}}^{J^k_j}\end{bmatrix}, \tag{42}$$

$$\underline{\mathscr{F}}^{B^{k,i}}_1 \triangleq \begin{bmatrix}-\underset{\sim}{\mathbf{I}}^{k,i}\cdot\hat{\boldsymbol{\alpha}}^{B^{k,i}}-\boldsymbol{\omega}^{B^{k,i}}\times\underset{\sim}{\mathbf{I}}^{B^{k,i}}\cdot\boldsymbol{\omega}^{B^{k,i}}+\mathbf{T}^{B^{k,i}}\\ -\underset{\sim}{\mathbf{M}}^{B^{k,i}}\cdot\hat{\mathbf{a}}^{B^{k,i}}+\mathbf{R}^{B^{k,i}}\end{bmatrix}\quad\text{and}$$

$$\underline{\mathscr{F}}^{J^k_j}_1 \triangleq \begin{bmatrix}-\underset{\sim}{\mathbf{I}}^{J^k_j}\cdot\hat{\boldsymbol{\alpha}}^{J^k_j}-\boldsymbol{\omega}^{J^k_j}\times\underset{\sim}{\mathbf{I}}^{J^k_j}\cdot\boldsymbol{\omega}^{J^k_j}+\mathbf{T}^{J^k_j}\\ -\underset{\sim}{\mathbf{M}}^{J^k_j}\cdot\hat{\mathbf{a}}^{J^k_j}+\mathbf{R}^{J^k_j}\end{bmatrix}, \tag{43}$$

$$\underline{\bar{\mathscr{A}}}^{B^{k,i}} \triangleq \begin{bmatrix}\bar{\boldsymbol{\alpha}}^{B^{k,i}}\\ \bar{\mathbf{a}}^{B^{k,i}}\end{bmatrix}\quad\text{and}\quad \underline{\bar{\mathscr{A}}}^{J^k_j} \triangleq \begin{bmatrix}\bar{\boldsymbol{\alpha}}^{J^k_j}\\ \bar{\mathbf{a}}^{J^k_j}\end{bmatrix}, \tag{44}$$

$$\underline{\mathscr{S}}^{B^{k,i}} \triangleq \begin{bmatrix}\underset{\sim}{\mathbf{U}} & \underset{\sim}{\boldsymbol{\xi}^{k,i}\times}\\ \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{U}}\end{bmatrix}\quad\text{and}\quad \underline{\mathscr{S}}^{J^k_j} \triangleq \begin{bmatrix}\underset{\sim}{\mathbf{U}} & \underset{\sim}{\boldsymbol{\gamma}^{J^k_j}\times}\\ \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{U}}\end{bmatrix}. \tag{45}$$

The terms $\mathbf{T}^P$ and $\mathbf{R}^P$ appearing in eqn (43) are the torque and force which line of action passes through P, and which together are equivalent to the set of all distance, contact and elastic forces (except those arising from geometric stiffness) acting on P.

In addition, from the corrective term matrix, $\underline{\mathscr{R}}^{B^k}_G$, of forces/torques which arise due to geometric stiffness and given by

$$\underline{\mathscr{R}}^{B^k}_G = \underline{K}^{B^k}_G\delta^{B^k} = \underline{K}^{B^k}_G\underline{\boldsymbol{\Phi}}^{B^k}_r q^{B^k}_r, \tag{46}$$

through matrix manipulation (Appendix), one can construct

$$\underline{\mathscr{F}}^{B^{k,i}}_G = \begin{bmatrix}\mathbf{T}^{B^{k,i}}_g\\ \mathbf{R}^{B^{k,i}}_g\end{bmatrix} = \Bigg([\underline{\hat{\boldsymbol{\Upsilon}}}^{B^{k,i}}_t\cdot\underline{\bar{\mathscr{A}}}^{B^k,J^k}+\underline{\hat{\mathscr{F}}}^{B^{k,i}}_{G_t}]$$

$$+\sum_{j\in Dist[k]}\underline{\check{\boldsymbol{\Upsilon}}}^{B^{k,i}_j}_t\cdot[\underline{\mathscr{I}}^{J^j_1}_3\cdot\underline{\bar{\mathscr{A}}}^{J^j_1}+\underline{\mathscr{F}}^{J^j_1}_3+(\underline{\mathscr{P}}^{J^j_1}_{J^j_1}\cdot\underline{\mathscr{F}}^{J^j_1})\underline{\mathscr{P}}^{J^j_1}_{J^j_1}]\Bigg)q^{B^k}_t, \tag{47}$$

where the vector quantities $\mathbf{T}^{B^{k,i}}_g$ and $\mathbf{R}^{B^{k,i}}_g$ are the torque/force pair which acts on grid i of

B^k due to geometric stiffness; $\underline{\hat{\Upsilon}}^{B^k,i}$ and $\underline{\check{\Upsilon}}^{B^k,i}$ are matrices of dyadic quantities associated with the geometric stiffness due to body B^k inertia loads and loads due to distal bodies, respectively; and $\underline{\mathscr{F}}^{J_i^k} = [\mathbf{T}^{J_i^k}, \mathbf{R}^{J_i^k}]^{\mathrm{T}}$.

Starting from the terminal bodies and working inward, the composite inertia values as well as the active force and inertia remainder term portions of the inertia forces are determined for the triangularized equations by recursively using the relationships

$$\underline{\mathscr{F}}_3^{B^k,i} \triangleq \begin{cases} \underline{\mathscr{F}}_1^{B^k,i} & \text{if} \quad i \neq J^{Dist[k]}, \\ \underline{\mathscr{F}}_3^{J_0^{Dist[k]}} & \text{if} \quad i = J^{Dist[k]}, \end{cases} \tag{48}$$

$$\underline{\mathscr{F}}_2^{J_{j-1}^k} \triangleq \underline{\mathscr{F}}_1^{J_{j-1}^k} + \underline{\mathscr{F}}_3^{J_j^k},$$

$$\underline{\mathscr{I}}_3^{B^k,i} \triangleq \underline{\mathscr{I}}_1^{B^k,i} + \begin{cases} \underline{\mathscr{I}}_1^{B^k,i} & \text{if} \quad i \neq J^{Dist[k]}, \\ \underline{\mathscr{I}}_3^{J_0^{Dist[k]}} & \text{if} \quad i = J^{Dist[k]}, \end{cases} \tag{49}$$

$$\underline{\mathscr{I}}_2^{J_{j-1}^k} \triangleq \underline{\mathscr{I}}_1^{J_{j-1}^k} + \underline{\mathscr{I}}_3^{J_j^k}, \tag{50}$$

Where $\underline{\mathscr{I}}_3^{B^k,J^k}$ and $\underline{\mathscr{F}}_3^{B^k,J^k}$ are defined as

$$\underline{\mathscr{I}}_3^{B^k,J^k} \triangleq \sum_i^{\nu^k} \Bigg(\underline{\mathscr{S}}^{B^k,i} \cdot \underline{\mathscr{I}}_3^{B^k,i} \cdot (\underline{\mathscr{S}}^{B^k,i})^{\mathrm{T}} + (\underline{\mathscr{S}}^{B^k,i} \cdot \underline{\mathscr{I}}_3^{B^k,i} \cdot \underline{\Phi}_s^{k,i}) \cdot \underline{\underline{\mathbf{g}}}_s^{B^k} + \underline{\mathscr{S}}^{B^k,i} \cdot \underline{\hat{\Upsilon}}_t^{B^k,i} q_t^{B^k} + \sum_{j \in Dist[k]} [\underline{\mathscr{S}}^{B^k,i} \cdot \underline{\check{\Upsilon}}_t^{B_j^k,i} q_t^{B^k} \cdot \underline{\mathscr{I}}_3^{B^k,J_1^j} \cdot (\underline{\mathscr{S}}^{B^k,j} + \underline{\Phi}_s^{k,j} \cdot \underline{\underline{\mathbf{g}}}_s^{B^k})] \Bigg), \tag{51}$$

and

$$\underline{\mathscr{F}}_3^{B^k,J^k} \triangleq \sum_i^{\nu^k} \Bigg(\underline{\mathscr{S}}^{B^k,i} \cdot \underline{\mathscr{F}}_3^{B^k,i} + (\underline{\mathscr{S}}^{B^k,i} \cdot \underline{\mathscr{I}}_3^{B^k,i} \cdot \underline{\Phi}_s^{k,i}) d_s^{B^k} + \underline{\mathscr{S}}^{B^k,i} \cdot \underline{\hat{\mathscr{F}}}_{G_t}^{B^k,i} q_t^{B^k} + \sum_{j \in Dist[k]} [\underline{\mathscr{S}}^{B^k,i} \cdot \underline{\check{\Upsilon}}_t^{B_j^k,i} q_t^{B^k} \cdot (\underline{\mathscr{F}}_3^{J_1^j} + \underline{\mathscr{I}}_3^{J_1^k} \cdot \underline{\Phi}_s^{k,j} d_s^{B^k})] \Bigg), \tag{52}$$

with

$$\mathscr{M}_{rs}^{B^k} \triangleq \left[\sum_i^{\nu^k} (\underline{\Phi}_r^{k,i})^{\mathrm{T}} \cdot \underline{\mathscr{I}}_3^{B^k,i} \cdot \underline{\Phi}_s^{k,i} \right] + \sum_{j \in Dist[k]} \left(\left[\sum_i^{\nu^k} (\underline{\Phi}_r^{k,i})^{\mathrm{T}} \cdot \underline{\check{\Upsilon}}_t^{B_j^k,i} \right] q_t^{B^k} \cdot \underline{\mathscr{I}}_3^{J_1^j} \cdot \underline{\Phi}_s^{B^k,j} \right), \tag{53}$$

and

$$\underline{d}^{B^k} = [d_1^{B^k}, \ldots, d_{\mu^{B^k}}^{B^k}]^{\mathrm{T}} \triangleq (\underline{\mathscr{M}}^{B^k})^{-1} \underline{c}^{B^k}, \tag{54}$$

where

$$c_s^{B^k} = \sum_i^{\nu^k} \left((\underline{\Phi}_s^{k,i})^{\mathrm{T}} \cdot \underline{\mathscr{F}}_3^{B^k,i} + (\underline{\Phi}_s^{k,i})^{\mathrm{T}} \cdot \underline{\hat{\mathscr{F}}}_{G_t}^{B^k,i} q_t^{B^k} \right) + \sum_{j \in Dist[k]} \left(\left[\sum_i^{\nu^k} (\underline{\Phi}_s^{k,i})^{\mathrm{T}} \cdot \underline{\check{\Upsilon}}_t^{B_j^k,i} \right] q_t^{B^k} \cdot \underline{\mathscr{F}}_3^{J_1^j} \right), \tag{55}$$

and

$$\underline{\mathbf{g}}^{B^k} = [\underline{\mathbf{g}}_1^{B^k}, \ldots, \underline{\mathbf{g}}_{\mu^{B^k}}^{B^k}]^{\mathrm{T}} \triangleq (\mathscr{M}^{B^k})^{-1}\underline{\mathbf{f}}_1^{B^k}, \ldots, \underline{\mathbf{f}}_{\mu^{B^k}}^{B^k}. \tag{56}$$

with

$$\underline{\mathbf{f}}_s^{B^k} \triangleq \left[\sum_i^{\nu^k} ((\underline{\mathbf{\Phi}}^{k,i})^{\mathrm{T}} \cdot \underline{\mathscr{I}}_3^{B^k,i} \cdot (\underline{\mathscr{S}}^{B^k,i})^{\mathrm{T}} + (\underline{\mathbf{\Phi}}_s^{k,i})^{\mathrm{T}} \cdot \underline{\hat{\mathbf{\Upsilon}}}_t^{B^k,i}) q_t^{B^k} \right.$$
$$\left. + \sum_{j \in Dist[k]} \left(\left[\sum_i^{\nu^k} (\underline{\mathbf{\Phi}}_s^{k,i})^{\mathrm{T}} \cdot \underline{\check{\mathbf{\Upsilon}}}_t^{B_j^k,i} \right] q_t^{B^k} \cdot \underline{\mathscr{I}}_3^{B^k,J_1^j} \cdot (\underline{\mathscr{S}}^{B^k,j})^{\mathrm{T}} \right) \right]. \tag{57}$$

Equations (51)–(57) contain summations of quantities associated with all grid points used in the finite element discretization of B^k. The spatial summation (integration) is only performed once, producing time invariant coefficients to the temporal quantities (Appendix). Substituting the time invariant coefficients and associated temporal quantities into these equations yields

$$\mathscr{M}_{rs}^{B^k} \triangleq C_{rs}^{11,k} + \sum_{j \in Dist[k]} \left[\left(\underline{\mathbf{\Phi}}_r^{k,j} + \begin{bmatrix} \mathbf{G}_{rt}^{1,B_j^k} \\ \mathbf{G}_{rt}^{2,B_j^k} \end{bmatrix} \right)^{\mathrm{T}} q_t^{B^k} \cdot \underline{\mathscr{I}}_3^{J_1^j} \cdot \underline{\mathbf{\Phi}}_s^{B^k,j} \right], \tag{58}$$

$$c_r^{B^k} = C_{rs}^{1,k} q_s^{B^k} - \{ C_{rl}^{2,k} \hat{\mathbf{a}}_l^{B^k,J^k} + [C_{rl}^{3,k} + C_{rsl}^{4,k} q_s^{B^k}] \hat{\boldsymbol{\alpha}}_l^{B^k,J^k}$$
$$+ [C_{rlm}^{5,k} + C_{rslm}^{6,k} q_s^{B^k}] \omega_l^{B^k,J^k} \omega_m^{B^k,J^k} + [C_{rsl}^{7,k} \omega_l^{B^k,J^k} + C_{rst}^{8,k} u_t^{B^k}] u_s^{B^k} \}$$
$$+ G_{st}^{3,B_j^k} q_t^{B^k} + \sum_{j \in Dist[k]} \left((\underline{\mathbf{\Phi}}_r^{k,j})^{\mathrm{T}} \cdot \underline{\mathscr{F}}_3^{J_1^j} + \begin{bmatrix} \mathbf{G}_{rt}^{1,B_j^k} \\ \mathbf{G}_{rt}^{2,B_j^k} \end{bmatrix}^{\mathrm{T}} q_t^{B^k} \cdot \underline{\mathscr{F}}_3^{J_1^j} \right), \tag{59}$$

and

$$\underline{\mathbf{f}}_r^{B^k} = \begin{bmatrix} C_r^{31,k} + C_{rs}^{32,k} q_s^{B^k} \\ C_r^{33,k} \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{rt}^{4,B_j^k} \\ \mathbf{G}_{rt}^{5,B_j^k} \end{bmatrix} q_t^{B^k}$$
$$+ \sum_{j \in Dist[k]} \left((\underline{\mathbf{\Phi}}_r^{k,j})^{\mathrm{T}} \cdot \underline{\mathscr{I}}_3^{J_1^k} \cdot (\underline{\mathscr{S}}^{B^k,j})^{\mathrm{T}} + \begin{bmatrix} \mathbf{G}_{rt}^{1,B_j^k} \\ \mathbf{G}_{rt}^{2,B_j^k} \end{bmatrix}^{\mathrm{T}} q_t^{B^k} \cdot \underline{\mathscr{I}}_3^{J_1^j} \cdot (\underline{\mathscr{S}}^{B^k,J_1^j})^{\mathrm{T}} \right). \tag{60}$$

Defining intermediate terms $\chi^{1,B^k}, \ldots, \chi^{7,B^k}$ and $\underline{\mathscr{G}}^{1,B^k}, \ldots, \underline{\mathscr{G}}^{8,B^k}$ as

$$\chi_l^{1,B^k} \triangleq [C_{sl}^{12,k} + C_{stl}^{13,k} u_t^{B^k}] q_s^{B^k} - \{ [C_{lm}^{14,k} + C_{slm}^{15,k} q_s^{B^k}] \hat{\mathbf{a}}_m^{B^k,J^k}$$
$$+ [C_{lm}^{16,k} + (C_{slm}^{17,k} + C_{slm}^{18,k} + C_{stlm}^{19,k} q_t^{B^k}) q_s^{B^k}] \hat{\boldsymbol{\alpha}}_m^{B^k,J^k}$$
$$+ [C_{lmn}^{20,k} + (C_{slmn}^{21,k} + C_{stlmn}^{22,k} q_t^{B^k}) q_s^{B^k}] \omega_m^{B^k,J^k} \omega_n^{B^k,J^k}$$
$$+ [C_{slm}^{23,k} + 2(C_{slm}^{17,k} + C_{stlm}^{19,k} q_t^{B^k})] \omega_m^{B^k,J^k} u_s^{B^k} + C_{stl}^{24,k} u_s^{B^k} u_t^{B^k} \}, \tag{61}$$

$$\chi_l^{2,B^k} = C_{sl}^{25,k} q_s^{B^k} - \{ C^{26,k} \hat{\mathbf{a}}_l^{B^k,J^k} + [C_{lm}^{27,k} + C_{slm}^{28,k} q_s^{B^k}] \hat{\boldsymbol{\alpha}}_m^{B^k,J^k}$$
$$+ [C_{lmn}^{29,k} + C_{slmn}^{30,k} q_s^{B^k}] \omega_m^{B^k,J^k} \omega_n^{B^k,J^k} + 2C_{slm}^{28,k} u_s^{B^k} \omega_m^{B^k,J^k} \}, \tag{62}$$

$$\chi_{rl}^{3,B^k} = C_{rl}^{31,k} + C_{rsl}^{32,k} q_s^{B^k}, \tag{63}$$

$$\chi_{rl}^{4,B^k} = C_{rl}^{33,k}, \tag{64}$$

$$\chi_{lm}^{5,B^k} = C_{lm}^{34,k} + [C_{slm}^{35,k} + C_{stlm}^{36,k} q_t^{B^k}] q_s^{B^k}, \tag{65}$$

$$\chi_{lm}^{6,B^k} = C_{lm}^{37,k} + C_{slm}^{38,k} q_s^{B^k}, \tag{66}$$

$$\chi_{lm}^{7,B^k} = C_{lm}^{39,k} + C_{slm}^{40,k} q_s^{B^k}, \tag{67}$$

and

$$\mathcal{G}_{tlm}^{1,B_j^k} = G_{tlm}^{6,B_j^k} + G_{stlm}^{7,B_j^k} q_s^{B^k}, \tag{68}$$

$$\mathcal{G}_{tlm}^{2,B_j^k} = G_{tlm}^{8,B_j^k} + G_{stlm}^{9,B_j^k} q_s^{B^k}, \tag{69}$$

$$\mathcal{G}_{tlm}^{3,B_j^k} = G_{tlm}^{10,B_j^k}, \tag{70}$$

$$\mathcal{G}_{tlm}^{4,B_j^k} = G_{tlm}^{11,B_j^k}, \tag{71}$$

$$\mathcal{G}_{tlm}^{5,B_j^k} = G_{tlm}^{12,B_j^k} + G_{stlm}^{13,B_j^k} q_s^{B^k}, \tag{72}$$

$$\mathcal{G}_{tlm}^{6,B_j^k} = G_{tlm}^{14,B_j^k} + G_{stlm}^{15,B_j^k} q_s^{B^k}, \tag{73}$$

$$\mathcal{G}_{tlm}^{7,B_j^k} = G_{tlm}^{16,B_j^k}, \tag{74}$$

$$\mathcal{G}_{tlm}^{8,B_j^k} = G_{tlm}^{17,B_j^k}, \tag{75}$$

$$\mathcal{G}_{stl}^{9,B^k} = G_{stl}^{18,B^k} + G_{rstl}^{19,B^k} q_r^{B^k}, \tag{76}$$

$$\mathcal{G}_{stl}^{10,B^k} = G_{stl}^{20,B^k}, \tag{77}$$

$\underline{\mathcal{I}}_3^{B^k,J^k}$ and $\underline{\mathcal{F}}_{t_3}^{B^k,J^k}$ may be written as

$$\underline{\mathcal{I}}_3^{B^k,J^k} = \begin{bmatrix} \chi^{5,B^k} & (\chi^{6,B^k})^{\mathrm{T}} \\ \chi^{6,B^k} & \chi^{7,B^k} \end{bmatrix} + \begin{bmatrix} \chi_r^{3,B^k} \\ \chi_r^{4,B^k} \end{bmatrix} \cdot \underline{\mathbf{g}}_r^k + \begin{bmatrix} \mathcal{G}_t^{1,B^k} & \mathcal{G}_t^{2,B^k} \\ \mathcal{G}_t^{3,B^k} & \mathcal{G}_t^{4,B^k} \end{bmatrix} q_t^{B^k}$$
$$+ \sum_{j \in Dist[k]} \left(\underline{\mathcal{S}}^{B^k,j} \cdot \underline{\mathcal{I}}_3^{J_1^j} \cdot (\underline{\mathcal{S}}^{B^k,j})^{\mathrm{T}} + (\underline{\mathcal{S}}^{B^k,j} \cdot \underline{\mathcal{I}}_3^{J_1^k} \cdot \underline{\mathbf{\Phi}}_s^{k,j}) \cdot \underline{\mathbf{g}}_s^{B^k} + \begin{bmatrix} \mathcal{G}_t^{5,B_j^k} & \mathcal{G}_t^{6,B_j^k} \\ \mathcal{G}_t^{7,B_j^k} & \mathcal{G}_t^{8,B_j^k} \end{bmatrix} q_t^{B^k} \right), \tag{78}$$

$$\underline{\mathcal{F}}_3^{B^k,J^k} \triangleq \begin{bmatrix} \chi^{1,B^k} \\ \chi^{2,B^k} \end{bmatrix} + \begin{bmatrix} \chi_r^{3,B^k} \\ \chi_r^{4,B^k} \end{bmatrix} d_r^{B^k} + \begin{bmatrix} \mathcal{G}_{st}^{9,B^k} \\ \mathcal{G}_{st}^{10,B^k} \end{bmatrix} \hat{A}_{G_s}^{B^k} q_t^{B^k} + \sum_{j \in Dist[k]} \left(\underline{\mathcal{S}}^{B^k,j} \cdot \underline{\mathcal{F}}_3^{J_1^k} \right.$$
$$\left. + (\underline{\mathcal{S}}^{B^k,j} \cdot \underline{\mathcal{I}}_3^{J_1^k} \cdot \underline{\mathbf{\Phi}}_s^{k,j}) d_s^{B^k} + \begin{bmatrix} \mathcal{G}_t^{5,B_j^k} & \mathcal{G}_t^{6,B_j^k} \\ \mathcal{G}_t^{7,B_j^k} & \mathcal{G}_t^{8,B_j^k} \end{bmatrix} q_t^{B^k} \cdot (\underline{\check{\mathcal{F}}}_G^{B_j^k} + \underline{\mathcal{I}}_3^{J_1^k} \cdot \underline{\mathbf{\Phi}}_r^{k,j} d_r^{B^k}) \right). \tag{79}$$

For the rigid body degrees of freedom the expressions for $\underline{\mathcal{I}}_3^{J^k}$ and $\underline{\mathcal{F}}_3^{J^k}$ are

$$\underline{\mathcal{I}}_3^{J_{j-1}^k} \triangleq \underline{\mathcal{I}}_1^{J_{j-1}^k} + \underline{\mathcal{S}}^{J_j^k} \cdot \underline{\mathcal{I}}_3^{J_j^k} \cdot (\underline{\mathcal{S}}^{J_j^k})^{\mathrm{T}} - \left(\frac{1}{(\underline{\mathcal{P}}^{J_j^k})^{\mathrm{T}} \cdot \underline{\mathcal{I}}_2^{J_j^k} \cdot \underline{\mathcal{P}}_{J_j^k}^{J_j^k}} \right)$$
$$\times [\underline{\mathcal{S}}^{J_j^k} \cdot \underline{\mathcal{I}}_3^{J_j^k} \cdot \underline{\mathcal{P}}_{J_j^k}^{J_j^k} \cdot (\underline{\mathcal{S}}^{J_j^k} \cdot \underline{\mathcal{I}}_3^{J_j^k} \cdot \underline{\mathcal{P}}_{J_j^k}^{J_j^k})^{\mathrm{T}}], \tag{80}$$

$$\underline{\mathcal{F}}_3^{J_{j-1}^k} \triangleq \underline{\mathcal{F}}_1^{J_{j-1}^k} + \underline{\mathcal{S}}^{J_j^k} \cdot \underline{\mathcal{F}}_3^{J_j^k} - \left(\frac{1}{(\underline{\mathcal{P}}_{J_j^k}^{J_j^k})^{\mathrm{T}} \cdot \underline{\mathcal{I}}_2^{J_j^k} \cdot \underline{\mathcal{P}}_{J_j^k}^{J_j^k}} \right) [\underline{\mathcal{S}}^{J_j^k} \cdot \underline{\mathcal{I}}_3^{J_j^k} \cdot \underline{\mathcal{P}}_{J_j^k}^{J_j^k} \cdot (\underline{\mathcal{P}}_{J_j^k}^{J_j^k})^{\mathrm{T}} \cdot \underline{\mathcal{F}}_3^{J_j^k}]. \tag{81}$$

The process is repeated, working recursively inward, to the lowest rigid body degree of freedom of the *base body*, at which time it is possible to write

$$\dot{u}_1^{J^1} = -\frac{(\underline{\mathscr{P}}_{J^1_1}^{J^1_1})^{\mathrm{T}} \cdot \underline{\mathscr{F}}_3^{J^1_1}}{(\underline{\mathscr{P}}_{J^1_1}^{J^1_1})^{\mathrm{T}} \cdot \underline{\mathscr{I}}_2^{J^1_1} \cdot \underline{\mathscr{P}}_{J^1_1}^{J^1_1}}, \tag{82}$$

providing the value of $\dot{u}_1^{J^1}$ in terms of known quantities.

(iii) *Recursive back-substitution.* This known value for $\dot{u}_1^{J^1}$ is used to start the back-substitution process for the determination of the remaining generalized speed derivatives. The recursive relationships used here are

$$\underline{\bar{\mathscr{A}}}^{J^k_j} = (\underline{\mathscr{S}}^{J^k_j})^{\mathrm{T}} \cdot \underline{\bar{\mathscr{A}}}^{J^k_{j-1}} + \underline{\mathscr{P}}_{J^k_j}^{J^k_j} \dot{u}_j^{J^k} \tag{83}$$

and

$$\dot{u}_j^{J^k} = \frac{-(\underline{\mathscr{P}}_{J^k_j}^{J^k_j})^{\mathrm{T}}}{(\underline{\mathscr{P}}_{J^k_j}^{J^k_j})^{\mathrm{T}} \cdot \underline{\mathscr{I}}_2^{J^k_j} \cdot \underline{\mathscr{P}}_{J^k_j}^{J^k_j}} \cdot [\underline{\mathscr{I}}_3^{J^k_j} \cdot (\underline{\mathscr{S}}^{J^k_j})^{\mathrm{T}} \cdot \underline{\bar{\mathscr{A}}}^{J^k_{j-1}} + \underline{\mathscr{F}}_3^{J^k_j}] \tag{84}$$

for rigid body degrees of freedom, and

$$\underline{\bar{\mathscr{A}}}^{B^{k,i}} = (\underline{\mathscr{S}}^{B^{k,i}})^{\mathrm{T}} \cdot \underline{\bar{\mathscr{A}}}^{B^k,J^k} + \underline{\Phi}_s^{k,i} \dot{u}_s^{B^k} \tag{85}$$

and

$$\dot{u}_s^{B^k} = d_s^{B^k} + \underline{\mathbf{g}}_s^{B^k} \cdot \underline{\bar{\mathscr{A}}}^{B^k,J^k} \tag{86}$$

for flexible body degrees of freedom. $\dot{u}_1^{J^1}$ is substituted into eqn (83) for the determination of $\underline{\bar{\mathscr{A}}}^{J^1_1}$, which is in turn used in eqn (84) for the determination of $\dot{u}_2^{J^1}$. This process is repeated, working recursively outward to determine the remaining joint accelerations by the appropriate application of eqns (83) and (84), and the state derivatives associated with the flexible degrees of freedom by use of eqns (85) and(86). The end result is that the rigid body degrees of freedom are uncoupled and the relative joint accelerations, $\dot{u}_j^{J^k}$ ($k = 1, \ldots, \mathscr{N}$ and $j = 1, \ldots, u^{J^k}$) are determined in $O(\mathscr{N})$ operations overall for a general tree structure. The state derivatives associated with the flexible degrees of freedom, $\dot{u}_s^{B^k}$ ($k = 1, \ldots, \mathscr{N}$ and $s = 1, \ldots, \mu^{B^k}$) are only coupled through the flexible degrees of freedom associated with B^k. This coupling manifests itself in $\underline{\mathscr{M}}^{B^k}$ in eqn (53). However, because the number of flexible degrees of freedom associated with a single body of a larger multibody system is often small in a global sense, the computational effort associated with the formation and inversion of these matrices is generally not the dominant factor in overall computational cost.

2.4. *Constrained systems*

The dynamical equations presented so far apply only to tree configurations which are not subject to either motion or configuration constraints. If a system has one or more constraints on its geometry or motion, then additional equations must be satisfied.

Consider an unconstrained multibody dynamical system possessing n degrees of freedom. The motion of the system is fully specified by the generalized speeds, $u_1, \ldots, u_n$, which are independent of each other. If the system is subjected to m conditions of the form

$$\Phi_i(q, t) = 0 \quad \text{and} \quad \Phi_i(q, \dot{q}, t) = 0 \quad (i = 1, \ldots, m) \tag{87}$$

for holonomic and nonholonomic constraints, respectively, then the motion of the system in a Newtonian reference frame is characterized by n generalized speeds $u_1, \ldots, u_n$ which are not independent of each other, but must satisfy m simple nonholonomic constraints, or holonomic constraints differentiated once with respect to time. Such constraint equations are of the form

$$\sum_{r=1}^{n} A_{rs}u_r + B_s = 0 \quad (s = 1, \ldots, m), \tag{88}$$

where A_{rs} and B_s are explicit functions of $q_1, \ldots, q_n$ and time t. As a consequence of the imposition of these constraints, the number of degrees of freedom of the system reduces from n to $p \triangleq n-m$.

The procedure presented here for the formulation of the equations of motion for systems subject to constraints is a variation on that presented by Park and Chiou (1988) and Park *et al.* (1989), and applied to an $O(n)$ approach in Anderson (1990).

To illustrate the procedure, the case of nonholonomic constraints will be considered. As an initial estimate of the generalized constraint forces, $\underline{\tilde{f}_c}$, required to enforce the constraint conditions, the generalized constraint forces are approximated as being proportional to the error in (88) and are given by

$$\underline{\tilde{f}_c} = \frac{1}{\varepsilon}(\underline{A}\,\underline{u} + \underline{B}), \tag{89}$$

where ε is a constant chosen by the analyst. This expression is, in turn, differentiated once with respect to time, yielding

$$\underline{\dot{\lambda}} = \frac{1}{\varepsilon}(\underline{A}\,\underline{\dot{u}} + \underline{\dot{A}}\,\underline{u} + \underline{\dot{B}}). \tag{90}$$

From the preceding discussion of unconstrained systems, it can be shown that the equations of motion for the constrained system can be written in the form

$$\underline{\dot{u}} + \underline{\Gamma}\,\underline{f_c} = \underline{\eta} \tag{91}$$

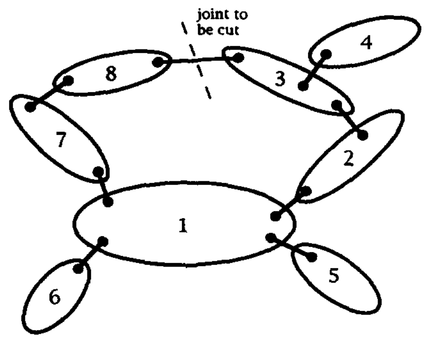
subject to the constraints

$$\underline{A}\underline{u} + \underline{\dot{A}}\underline{u} + \underline{\dot{B}} = \underline{0}. \tag{92}$$

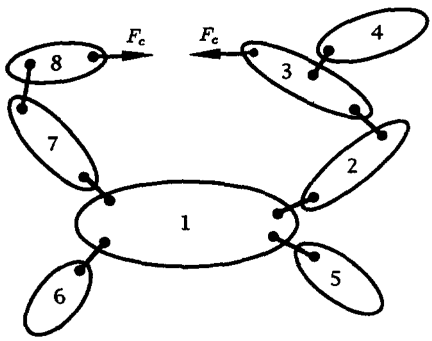
Solving (91) for $\underline{\dot{u}}$ and substituting this into (90) yields

$$\varepsilon\underline{\dot{f}_c} + \underline{A}\,\underline{\Gamma}\,\underline{f_c} = \underline{A}\,\underline{\eta} + \underline{b}. \tag{93}$$

The solution of this ordinary differential equation in $\underline{f_c}$ decays to the constraint load measure number values. When constraints are present, the procedure follows much the same course as the basic algorithm presented in the previous section, but now the presence of the unknown constraint force measure numbers in the equations of motion must be considered. For instance, in Fig. 3(a), body 3 closes the loop in the system through its connection with body 8. This algorithm, like most others, requires that the system be a tree structure. This necessitates that closed loops be cut at the joints connecting appropriate bodies so that the required open loop structure is produced, Fig. 3(b). The constraint conditions which insure closure of the loops are then enforced through the addition of constraint forces and moments of proper magnitude and direction, applied at the connection points. In general, neither the magnitudes nor the direction of these constraint forces and moments are constant, which necessitates expressing these loads as the vector sum of their components in some meaningful basis, say one fixed in the Newtonian frame N. In the basic algorithm, the set of all distance and contact forces acting on $P^{k,i}$ are given by the reaction $\mathbf{R}^{B^{k,i}}$ and torque $\mathbf{T}^{B^{k,i}}$. The presence of the generalized constraint forces $f_{c_1}, \ldots, f_{c_m}$ now



(A) CLOSED LOOP SYSTEM



(B) CORRESPONDING OPEN LOOP SYSTEM

Fig. 3. Tree systems associated with closed loop systems.

applied to enforce the m scalar constraint conditions should be included in these quantities. However, all forces and moments acting on B^k, except for the generalized constraint force, are known. So, it is highly desirable to keep the unknown generalized constraint forces and the known applied forces segregated. To this end, the resultant applied forces and torques are given by

$$\mathbf{R}^{B^k,h}_{\text{total}} = \mathbf{R}^{B^k,h} + \mathbf{R}^{B^k,h}_c \quad \text{and} \quad \mathbf{T}^{B^k,h}_{\text{total}} = \mathbf{T}^{B^k,h} + \mathbf{T}^{B^k,h}_c, \tag{94}$$

respectively, from which we define

$$\underline{\mathscr{F}^{B^k,h}_{c_1}} \triangleq \begin{bmatrix} \mathbf{T}^{B^k,h}_c \\ \mathbf{R}^{B^k,h}_c \end{bmatrix}, \tag{95}$$

where h is the point of B^k through which a constraint is applied.

Defining the quantities $d^{B^k}_c$, $\underline{\mathscr{F}^{B^k,J^k}_{c_3}}$ and $\underline{\mathscr{F}^{J^k_{j-1}}_{c_3}}$ as

$$\underline{d^{B^k}_c} \triangleq (\underline{\mathscr{M}}^{B^k})^{-1}\left(\sum_h (\underline{\mathbf{\Phi}}^{k,h}_r)^{\mathrm{T}} \cdot \underline{\mathscr{F}^{B^k,h}_{c_3}} + \begin{bmatrix} \mathbf{G}^{1,B^k_h}_{rt} \\ \mathbf{G}^{2,B^k_h}_{rt} \end{bmatrix}^{\mathrm{T}} q^{B^k}_t \cdot \underline{\mathscr{F}^{B^k,h}_{c_3}} \right.$$
$$\left. + \sum_{j \in Dist[k]} \left(\begin{bmatrix} \mathbf{G}^{1,B^k_j}_{rt} \\ \mathbf{G}^{2,B^k_j}_{rt} \end{bmatrix}^{\mathrm{T}} q^{B^k}_t \cdot \underline{\mathscr{F}^{B^k,J^j_0}_{c_3}} \right)\right), \tag{96}$$

$$\underline{\mathscr{F}}_{c_3}^{B^k,J^k} \triangleq \sum_h \left(\underline{\mathscr{S}}^{B^k,h} \cdot \underline{\mathscr{F}}_{c_1}^{B^k,h} + \underline{\mathscr{S}}^{B^k,h} \cdot \underline{\mathscr{I}}_2^{B^k,h} \cdot \underline{\Phi}_r^{k,h} d_{c_r}^{B^k} + \begin{bmatrix} \mathscr{G}_t^{5,B^k,h} & \mathscr{G}_t^{6,B^k,h} \\ \mathscr{G}_t^{7,B^k,h} & \mathscr{G}_t^{8,B^k,h} \end{bmatrix} q_t^{B^k} \cdot \underline{\mathscr{F}}_{c_1}^{B^k,h} \right)$$

$$+ \sum_{j \in Dist[k]} \left(\begin{bmatrix} \mathscr{G}_t^{5,B_j^k} & \mathscr{G}_t^{6,B_j^k} \\ \mathscr{G}_t^{7,B_j^k} & \mathscr{G}_t^{8,B_j^k} \end{bmatrix} q_t^{B^k} \cdot \underline{\mathscr{F}}_{c_3}^{J_0^j} \right), \quad (97)$$

$$\underline{\mathscr{F}}_{c_3}^{J_{j-1}^k} \triangleq \underline{\mathscr{F}}_{c_1}^{J_{j-1}^k} + \underline{\mathscr{S}}^{J_j^k} \cdot \underline{\mathscr{F}}_{c_3}^{J_j^k} - \left(\frac{1}{(\underline{\mathscr{P}}_{J_j^k}^{J_j^k})^{\mathrm{T}} \cdot \underline{\mathscr{I}}_2^{J_j^k} \cdot \underline{\mathscr{P}}_{J_j^k}^{J_j^k}} \right) [\underline{\mathscr{S}}^{J_j^k} \cdot \underline{\mathscr{I}}_3^{J_j^k} \cdot \underline{\mathscr{P}}_{J_j^k}^{J_j^k} \cdot (\underline{\mathscr{P}}_{J_j^k}^{J_j^k})^{\mathrm{T}} \cdot \underline{\mathscr{F}}_{c_3}^{J_j^k}], \quad (98)$$

the state derivatives for the constrained system are given by

$$\dot{u}_j^{J^k} = \frac{-(\underline{\mathscr{P}}_{J_j^k}^{J_j^k})^{\mathrm{T}}}{(\underline{\mathscr{P}}_{J_j^k}^{J_j^k})^{\mathrm{T}} \cdot \underline{\mathscr{I}}_2^{J_j^k} \cdot \underline{\mathscr{P}}_{J_j^k}^{J_j^k}} \cdot [\underline{\mathscr{I}}_3^{J_j^k} \cdot (\underline{\mathscr{S}}^{J_j^k})^{\mathrm{T}} \cdot \underline{\bar{\mathscr{A}}}^{J_{j-1}^k} + \underline{\mathscr{F}}_3^{J_j^k} + \underline{\mathscr{F}}_{c_3}^{J_j^k}] \quad (99)$$

and

$$\underline{\dot{u}}^{B^k} = \underline{d}^{B^k} + \underline{d}_c^{B^k} + \underline{\mathbf{g}}^{B^k} \cdot \underline{\bar{\mathscr{A}}}^{B^k,J^k}. \quad (100)$$

Expressing $\underline{\dot{u}}$ as

$$\underline{\dot{u}} = \underline{\eta} + \underline{\zeta}, \quad (101)$$

where $\underline{\zeta}$ is that portion of $\underline{\dot{u}}$ which is explicit in the constraint load measure numbers and $\underline{\eta}$ is all else, $\underline{\eta}$ is simply equal to $\underline{\dot{u}}$ for the identical system when no constraints are imposed and is given by eqns (83)–(85). In a similar manner, the elements of $\underline{\zeta}$ are given by the relations

$$\zeta_j^{J^k} = \frac{-(\underline{\mathscr{P}}_{J_j^k}^{J_j^k})^{\mathrm{T}}}{(\underline{\mathscr{P}}_{J_j^k}^{J_j^k})^{\mathrm{T}} \cdot \underline{\mathscr{I}}_2^{J_j^k} \cdot \underline{\mathscr{P}}_{J_j^k}^{J_j^k}} \cdot [\underline{\mathscr{I}}_3^{J_j^k} \cdot (\underline{\mathscr{S}}^{J_j^k})^{\mathrm{T}} \cdot \underline{\bar{\zeta}}^{J_j^k} + \underline{\mathscr{F}}_{c_3}^{J_j^k}] \quad (102)$$

$$= -\sum_{i=1}^{m} \Gamma_i^{J_j^k} f_{c_i}^{J_j^k}, \quad (103)$$

where

$$\underline{\bar{\zeta}}^{J_j^k} = (\underline{\mathscr{S}}^{J_j^k})^{\mathrm{T}} \cdot \underline{\bar{\zeta}}^{J_{j-1}^k} + \underline{\mathscr{P}}_{J_j^k}^{J_j^k} \zeta_j^{J^k} \quad (104)$$

and

$$\zeta_j^{B^k} = d_{c_j}^{B^k} + \underline{\mathbf{g}}_j^{B^k} \cdot \underline{\bar{\zeta}}^{B^k,J^k} \quad (105)$$

$$= -\sum_{i=1}^{m} \Gamma_{ji}^{B^k} f_{c_i}, \quad (106)$$

where

$$\underline{\bar{\zeta}}^{B^k,j} \triangleq (\underline{\mathscr{S}}^{B^k,j})^{\mathrm{T}} \cdot \underline{\bar{\zeta}}^{B^k,J^k} + \underline{\Phi}_s^{k,j} \zeta_s^{B^k}. \quad (107)$$

So, the equations of motion for the constrained system may be written as

$$\underline{\dot{u}}+\underline{\Gamma}\,\underline{f_c} = \underline{\eta}, \tag{108}$$

where $\underline{\Gamma}$ is defined by (103) and (106), and $\underline{f_c}$ is obtained from eqn (93). The state derivatives are then calculated from eqn (108), with the total procedure requiring approximately $O(n+m^3)$ operations.

3. NUMERICAL EXAMPLES

Three examples are provided to validate the formulation and offer evidence of the improved performance possible when using an $O(n)$ formulation for the analysis of systems containing a large number of bodies.

3.1. *Slider–crank mechanism*

The slider–crank mechanism is shown in Fig. 4. The crank A is rigid with the distance from O to P being L_a, and moves with constant angular speed Ω. Connecting beam B, from P to Q, is of uniform circular cross-section, has length L_b, and is elastic. The sliding block C is connected to B at Q and is constrained to motion along the x-axis only. The admissible functions used in modeling the flexibility for B were obtained from finite element analysis using MSC NASTRAN "cbar" beam elements. In each of the three cases considered, all available modes were used which were associated with motion of B in the x–y plane. Deflections are measured between the centerlines of the deformed and undeformed beam B at its midpoint. The properties of B used in this example are:

L_a	$= 6.0$ in,
L_b	$= 12.0$ in,
Young's Modulus B	$= 3.0\times10^7$ lbf in^{-2},
Diameter of B	$= 0.25$ in,
Mass B	$= 0.1667$ lbm,
Mass C	$= 0$ lbm,
Ω	$= 125.6$ rad sec^{-1}.

The initial position of the mechanism is with P in (x, y) coordinates $(0, 6)$ and no deformation of B.

The three cases analysed are:

Case 1. B is composed of a single element,
Case 2. B is composed of two equal length elements,
Case 3. B is composed of three equal length elements.

The results are shown in Fig. 5 and agree almost perfectly with those obtained by Lee (1988). In his work, Lee used a formulation similar to that in Singh *et al.* (1984) and imposed constraints through the implementation of the method presented by Wampler (1985).

3.2. *Cantilever beam with prescribed base motion*

The system analysed here consists of a cantilever beam attached to a base with prescribed angular velocity—an example used as a benchmark test by a number of authors

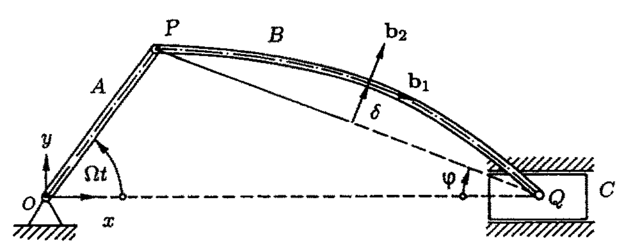


Fig. 4. Slider–crank mechanism.

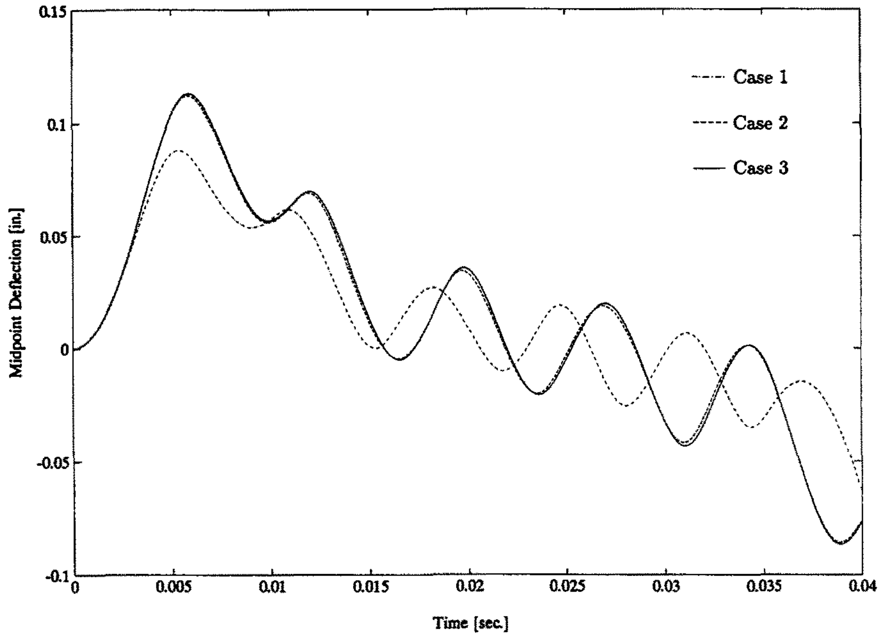


Fig. 5. Transverse deflection of slider–crank connecting rod.

(Kane *et al.*, 1987; Ryan, 1987; Banerjee and Dickens, 1990; Wallrapp and Schwertassek, 1991). The generalized coordinates used in this problem are the four modal coordinates and the angular velocity of the base is given by

$$\omega(t) = \begin{cases} \frac{2}{5}\left[t - \frac{7.5}{\pi}\sin\left(\frac{\pi t}{7.5}\right)\right] & \text{if} \quad t < 15\text{ sec}, \\ 6.0\text{ (rad sec}^{-1}) & \text{if} \quad t \geqslant 15\text{ sec}. \end{cases} \tag{109}$$

The beam geometry and properties which appear in this example were those used by Ryan (1987), as well as Wallrapp and Schwertassek (1991), and are:

Beam length	$= 10.0$ m,
Cross-sectional area	$= 0.0004$ m^2,
Area moment of inertia	$= 2.0 \times 10^{-7}$ m^4,
Young's Modulus	$= 7.0 \times 10^{10}$ N m^{-2},
Shear modulus	$= 3.0 \times 10^{10}$ N m^{-2},
Mass	$= 12$ kg.

Figure 6 shows the tip deflection of the beam central axis from its undeformed position during the course of a 20 second simulation. The results agree well with those obtained using the formulation presented in Kane *et al.* (1987).

3.3. $(n/2)$-*body chain with constrained ends*

The system consists of a chain of rigid bodies with each end of the chain connected to a point fixed in an inertial reference frame. The $n/2$ bodies of the chain are connected to each other and to points fixed in the inertial frame by two degree of freedom Hooke's joints (Fig. 7).

If n represents the number of degrees of freedom associated with the unconstrained system, then the system has $n/2$ bodies, $(n/2)+1$ joints, and the actual number of system degrees of freedom is $p = n-2$. The motion of the system was simulated for $n = 4$, 12, 20, 28, 36 and 44.

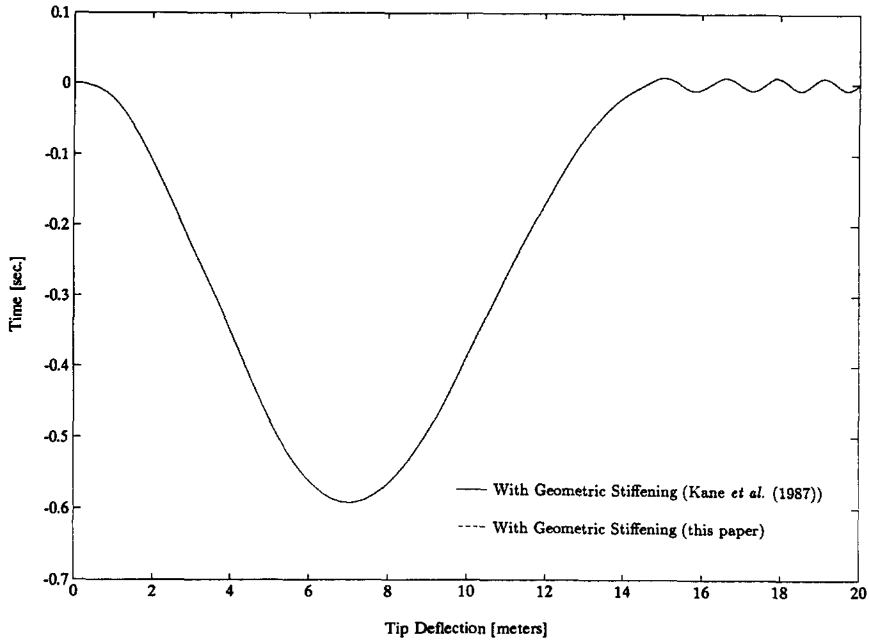


Fig. 6. Tip deflection of rotating cantilever beam.

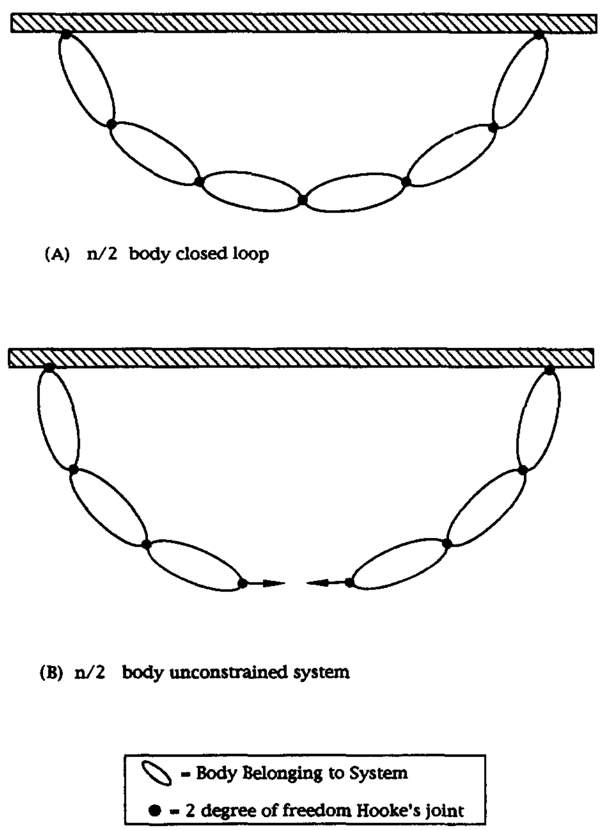


Fig. 7. Closed loop chain schematic.

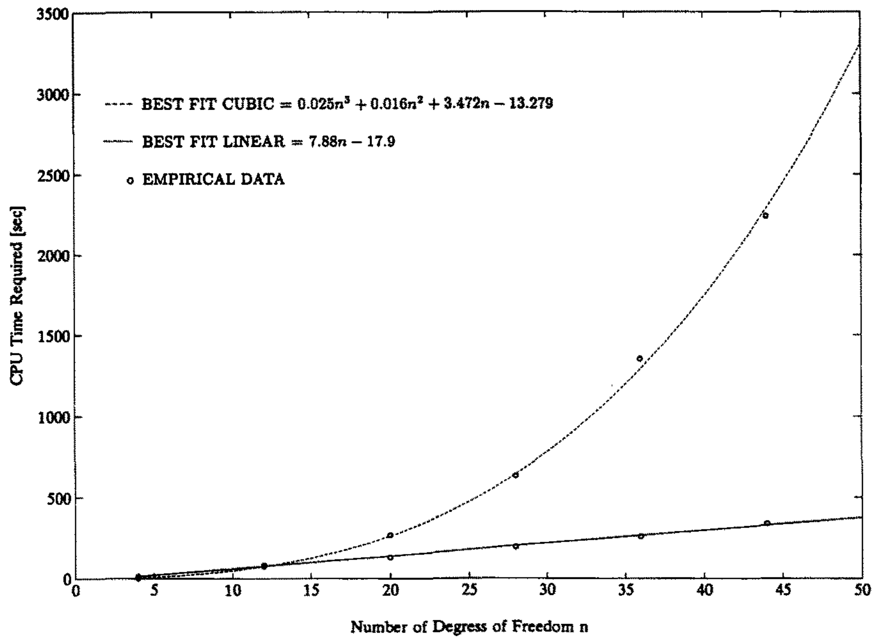


Fig. 8. CPU time as a function of n.

Results obtained from simulation codes produced using the $O(n)$ algorithm presented in this paper were verified through direct comparison with results obtained for identical systems using simulation codes written from equations of motion derived using the *standard* Kane's method as it is presented in Kane and Levinson (1985), producing an $O(n^3)$ procedure. In this standard formulation, the constraint relations were enforced through the technique presented by Wamper (1985).

The simulation results obtained by each of the two formulations were effectively identical. However, as indicated in Fig. 8, the CPU times required in performing a desired 1.0 s simulation differ markedly for each of these formulations when n is large. It is readily seen from the figure that substantial saving in computer time and associated cost are possible when using an $O(n)$ formulation relative to more conventional $O(n^3)$ formulations.

4. CONCLUDING REMARKS

A general formulation is presented for the analysis of transient response of multibody systems with flexible members. The method uses admissible shape functions derived from finite element modeling of the component members and thus allows the modeling of flexibility for general bodies. The formulation treats the general case of coupled large rigid body displacements, linear elastic deformation, and includes a first order representation geometric stiffening effects. Not addressed here are the difficulties associated with the inclusion of additional geometric stiffening terms which further improve simulation accuracy, or those in selecting the *best* set of admissible functions.

The formulation presented applies to systems involving three-dimensional motions, which may be comprised of any joint type which can be modeled as a series of revolute and prismatic joints. The equations of motion produced in this manner are uncoupled in the rigid body degrees of freedom, with coupling only existing between the flexible degrees of freedom associated with the individual flexible bodies. Furthermore, the equations are generated in a form which exploits the coarse grain concurrency of the mathematical model to the maximum degree. Thus, the computer simulation code produced from this type of formulation is particularly well suited for application to some forms of parallel computers.

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APPENDIX: DETERMINATION OF TIME INVARIANT COEFFICIENTS

Summations are carried out over all repeated indices ($h = 1,\ldots,6$; $j \in Dist[k]$; $l,m,n,o,p,u,v = 1,2,3$; $q = 1,\ldots,21$; $r,s,t = 1,\ldots,\mu^{B^k}$) and ε_{lmn} is the cyclic perturbation operator.

$$\begin{bmatrix} \mathbf{T}_r^{k,i} \\ \mathbf{f}_r^{k,i} \end{bmatrix} = \underline{\mathbf{F}}^{k,i}, \quad \text{where} \quad \mathbf{F}_{sh}^{k,i} = \underline{K}_{hr}^{B^k,i}\underline{\Phi}_{sr}^{B^k,i}, \tag{A1}$$

where $\underline{K}_{hr}^{B^k,i}$ is the rth element of the row of the body k elastic stiffness matrix associated with the hth degree of freedom of the ith grid.

$$C_{rs}^{1,k} = \sum_{1}^{\nu^k} (\phi_{rl}^{k,i}\mathbf{f}_{sl}^{k,i} + \psi_{rl}^{k,i}\mathbf{T}_{sl}^{k,i}) \tag{A2}$$

$$C_{rl}^{2,k} = \sum_{i=1}^{\nu^k} (\phi_{rl}^{k,i} m^{B^k,i}), \tag{A3}$$

$$C_{rl}^{3,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\rho_m^{k,i}\phi_{rn}^{k,i} m^{B^k,i} + \psi_{rm}^{k,i} I_{lm}^{B^k,i}), \tag{A4}$$

$$C_{rsl}^{4,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\phi_{sm}^{k,i}\phi_{rn}^{k,i} m^{B^k,i}), \tag{A5}$$

$$C_{rlo}^{5,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{mop}\rho_p^{k,i}\phi_{rn}^{k,i} m^{B^k,i} + \varepsilon_{mln}\psi_{rm}^{k,i} I_{no}^{B^k,i}), \tag{A6}$$

$$C_{rslo}^{6,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{mln}\varepsilon_{nop}\phi_{rm}^{k,i}\phi_{sp}^{k,i} m^{B^k,i}), \tag{A7}$$

$$C_{rsl}^{7,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\phi_{sm}^{k,i}\phi_{rn}^{k,i} m^{B^k,i} + \varepsilon_{mln}\psi_{ro}^{k,i}\psi_{sn}^{k,i} I_{om}^{B^k,i} + \varepsilon_{mln}\psi_{rm}^{k,i}\psi_{so}^{k,i} I_{no}^{B^k,i} + \varepsilon_{omn}\psi_{ro}^{k,i}\psi_{sm}^{k,i} I_{nl}^{B^k,i}), \tag{A8}$$

$$C_{rst}^{8,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\psi_{rl}^{k,i}\psi_{sm}^{k,i}\psi_{to}^{k,i} I_{no}^{B^k,i}), \tag{A9}$$

$$C_{rm}^{9,k} = \sum_{i=1}^{\nu^k} (\psi_{rl}^{k,i}\delta_{lm} m^{B^k,i}), \quad \delta_{lm} = \text{Kronecker Delta}, \tag{A10}$$

$$C_{rm}^{10,k} = \sum_{i=1}^{\nu^k} (\psi_{rl}^{k,i} I_{lm}^{B^k,i}), \tag{A11}$$

$$C_{rs}^{11,k} = \sum_{i=1}^{\nu^k} (\phi_{rl}^{k,i}\phi_{sm}^{k,i}\delta_{lm} m^{B^k,i} + \psi_{rl}^{k,i}\psi_{sm}^{k,i} I_{lm}^{B^k,i}), \tag{A12}$$

$$C_{sl}^{12,k} = \sum_{i=1}^{\nu^k} (T_{sl}^{k,i} + \varepsilon_{lmn}\rho_m^{B^k,i}\mathbf{f}_{sn}^{B^k,i}), \tag{A13}$$

$$C_{stl}^{13,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\phi_{sm}^{B^k,i}\mathbf{f}_{tn}^{B^k,i}), \tag{A14}$$

$$C_{ln}^{14,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\rho_m^{B^k,i} m^{B^k,i}), \tag{A15}$$

$$C_{sln}^{15,k} = \varepsilon_{lmn} \sum_{i=1}^{\nu^k} (\phi_{sm}^{B^k,i} m^{B^k,i}), \tag{A16}$$

$$C_{lm}^{16,k} = \sum_{i=1}^{\nu^k} (I_{lm}^{B^k,i} + \varepsilon_{lmn}\varepsilon_{nop}\rho_o^{B^k,i}\rho_p^{B^k,i} m^{B^k,i}), \tag{A17}$$

$$C_{slo}^{17,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{nop}\rho_m^{B^k,i}\phi_{sp}^{B^k,i}), \tag{A18}$$

$$C_{slo}^{18,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{nop}\rho_p^{B^k,i}\phi_{sm}^{B^k,i} m^{B^k,i}), \tag{A19}$$

$$C_{stlo}^{19,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{nop}\phi_{sm}^{B^k,i}\phi_{tp}^{B^k,i} m^{B^k,i}), \tag{A20}$$

$$C_{lmo}^{20,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn} I_{no}^{B^k,i} + \varepsilon_{lqn}\varepsilon_{nmp}\varepsilon_{pou}\rho_v^{B^k,i}\rho_u^{B^k,i} m^{B^k,i}), \tag{A21}$$

$$C_{slov}^{21,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{nop}\varepsilon_{pvu}[\rho_m^{B^k,i}\phi_{su}^{B^k,i} m^{B^k,i} + \rho_u^{B^k,i}\phi_{sm}^{B^k,i} m^{B^k,i}]), \tag{A22}$$

$$C_{stlov}^{22,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{nop}\varepsilon_{pvu}\phi_{sm}^{B^k,i}\phi_{tu}^{B^k,i} m^{B^k,i}), \tag{A23}$$

$$C_{sln}^{23,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{mno}\psi_{so}^{k,i} I_{lm}^{B^k,i} + \varepsilon_{nlm}\psi_{so}^{k,i} I_{mo}^{B^k,i} + \varepsilon_{nlm}\psi_{sm}^{k,i} I_{ml}^{B^k,i}), \tag{A24}$$

$$C_{stl}^{24,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\psi_{sm}^{k,i}\psi_{to}^{k,i} I_{no}^{B^k,i}), \tag{A25}$$

$$C_{sl}^{25,k} = \sum_{i=1}^{\nu^k} (\mathbf{f}_{sl}^{k,i}) = \sum_{i=1}^{\nu^k} (\underline{K}_{lr}^{B^k,i}\underline{\Phi}_{sr}^{B^k,i}). \tag{A26}$$

$$C_{sl}^{26,k} = \sum_{i=1}^{\nu^k} (m^{B^k,i}) = m^{B^k}, \tag{A27}$$

$$C_{lm}^{27,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\rho_n^{B^k,i} m^{B^k,i}), \tag{A28}$$

$$C_{slm}^{28,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\phi_n^{k,i} m^{B^k,i}), \tag{A29}$$

$$C_{lmo}^{29,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{nop}\rho_p^{k,i} m^{B^k,i}), \tag{A30}$$

$$C_{slmo}^{30,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{nop}\phi_{sp}^{k,i} m^{B^k,i}), \tag{A31}$$

$$C_{sl}^{31,k} = \sum_{i=1}^{\nu^k} (I^{B^k,i}\psi_{sm}^{k,i} + \varepsilon_{lmn}\rho_m^{k,i}\phi_{sn}^{k,i} m^{B^k,i}), \tag{A32}$$

$$C_{stl}^{32,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\phi_{sm}^{k,i}\phi_{tn}^{k,i} m^{B^k,i}), \tag{A33}$$

$$C_{sl}^{33,k} = \sum_{i=1}^{\nu^k} (\phi_{sl}^{k,i} m^{B^k,i}), \tag{A34}$$

$$C_{lv}^{34,k} = \sum_{i=1}^{\nu^k} (I_{lv}^{B^k,i} + \varepsilon_{lmn}\varepsilon_{nop}\delta_{ov}\phi_{sm}^{k,i}\rho_p^{k,i} m^{B^k,i}), \tag{A35}$$

$$C_{slv}^{35,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{nop}\delta_{ov}(\phi_{sp}^{k,i}\rho_m^{k,i} + \phi_{sm}^{k,i}\rho_p^{k,i}) m^{B^k,i}), \tag{A36}$$

$$C_{stlv}^{36,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lmn}\varepsilon_{nop}\delta_{ov}\phi_{sm}^{k,i}\phi_{tp}^{k,i}\rho_p^{k,i} m^{B^k,i}), \tag{A37}$$

$$C_{lm}^{37,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lon}\delta_{nm}\rho_o^{k,i} m^{B^k,i}), \tag{A38}$$

$$C_{slm}^{38,k} = \sum_{i=1}^{\nu^k} (\varepsilon_{lon}\delta_{nm}\phi_{so}^{k,i} m^{B^k,i}). \tag{A39}$$

The following are all terms associated with geometric stiffness. The equations hold if the matrices $\underline{\hat{K}}_{G_q}^{B^k,i}$ and $\underline{\breve{K}}_{G_{jh}}^{B^k,i}$ represent the six rows of $\underline{\hat{K}}_{G_q}^{B^k}$ and $\underline{\breve{K}}_{G_{jh}}^{B^k}$, respectively, which are associated with the six degrees of freedom of ith grid, and in the order;

row l corresponds to rotation in local direction l, $(l = 1, 2, 3)$,
row $l+3$ corresponds to translation in local direction l, $(l = 1, 2, 3)$.

$$D_{tlm}^{1,k,i} = (\underline{\hat{K}}_{G_{(m+3)lv}}^{B^k,i} + \underline{\hat{K}}_{G_{(m+12)lv}}^{B^k,i})\Phi_{tv}, \tag{A40}$$

$$D_{tlm}^{2,k,i} = \underline{\hat{K}}_{G_{mlv}}^{B^k,i}\Phi_{tv}, \tag{A41}$$

$$D_{tlm}^{3,k,i} = (\underline{\hat{K}}_{G_{(m+3)(l+3)v}}^{B^k,i} + \underline{\hat{K}}_{G_{(m+12)(l+3)v}}^{B^k,i})\Phi_{tv}, \tag{A42}$$

$$D_{tlm}^{4,k,i} = \underline{\hat{K}}_{G_{m(l+3)v}}^{B^k,i}\Phi_{tv}, \tag{A43}$$

$$G_{rsl}^{1,B_j^k} = (\Phi_{rv}^k)^{\mathrm{T}}\underline{\check{K}}_{G_{jlvw}}^{B^k}\Phi_{sw}^k, \tag{A44}$$

$$G_{rsl}^{2,B_j^k} = (\Phi_{rv}^k)^{\mathrm{T}}\underline{\check{K}}_{G_{j(l+3)vw}}^{B^k}\Phi_{sw}^k, \tag{A45}$$

$$G_{qrsl}^{3,B^k} = (\Phi_{rv}^k)^{\mathrm{T}}\underline{\hat{K}}_{G_{qvw}}^{B^k}\Phi_{sw}^k, \tag{A46}$$

$$G_{rs}^{4,B^k} = \sum_{i=1}^{\nu^k}(\phi_{rl}^{k,i}D_{tlm}^{1,k} + \psi_{rl}^{k,i}D_{tlm}^{3,k,i}), \tag{A47}$$

$$G_{rlm}^{5,B^k} = \sum_{i=1}^{\nu^k}(\phi_{rl}^{k,i}D_{tlm}^{2,k,i} + \psi_{rl}^{k,i}D_{tlm}^{4,k,i}), \tag{A48}$$

$$G_{tlo}^{6,B^k} = \sum_{i=1}^{\nu^k}(D_{tlo}^{1,k,i} + \varepsilon_{lmn}\rho_m^{k,i}D_{tno}^{3,k,i}), \tag{A49}$$

$$G_{tslo}^{7,B^k} = \sum_{i=1}^{\nu^k}(\varepsilon_{lmn}\phi_{sm}^{k,i}D_{tno}^{3,k,i}), \tag{A50}$$

$$G_{tlo}^{8,B^k} = \sum_{i=1}^{\nu^k}(D_{tlo}^{2,k,i} + \varepsilon_{lmn}\rho_m^{k,i}D_{tno}^{4,k,i}), \tag{A51}$$

$$G_{tslo}^{9,B^k} = \sum_{i=1}^{\nu^k}(\varepsilon_{lmn}\phi_{sm}^{k,i}D_{tno}^{4,k,i}), \tag{A52}$$

$$G_{tlm}^{10,B^k} = \sum_{i=1}^{\nu^k}(D_{tlm}^{3,k,i}), \tag{A53}$$

$$G_{tlm}^{11,B^k} = \sum_{i=1}^{\nu^k}(D_{tlm}^{4,k,i}), \tag{A54}$$

$$G_{slo}^{12,B_j^k} = \sum_{i=1}^{\nu^k}(\underline{\check{K}}_{G_{jolv}}^{B^k,i}\Phi_{sv}^k + \varepsilon_{lmn}\rho_m^{k,i}\underline{\check{K}}_{G_{jo(n+3)v}}^{B^k,i}\Phi_{sv}^k), \tag{A55}$$

$$G_{stlo}^{13,B_j^k} = \sum_{i=1}^{\nu^k}(\varepsilon_{lmn}\phi_{tm}^{k,i}\underline{\check{K}}_{G_{jo(n+3)v}}^{B^k,i}\Phi_{sv}^k), \tag{A56}$$

$$G_{slo}^{14,B_j^k} = \sum_{i=1}^{\nu^k}(\underline{\check{K}}_{G_{jolv}}^{B^k,i}\Phi_{sv}^k + \varepsilon_{lmn}\rho_m^{k,i}\underline{\check{K}}_{G_{j(o+3)(n+3)v}}^{B^k,i}\Phi_{sv}^k), \tag{A57}$$

$$G_{stlo}^{15,B_j^k} = \sum_{i=1}^{\nu^k}(\varepsilon_{lmn}\phi_{tm}^{k,i}\underline{\check{K}}_{G_{j(o+3)(n+3)v}}^{B^k,i}\Phi_{sv}^k), \tag{A58}$$

$$G_{slm}^{16,B_j^k} = \sum_{i=1}^{\nu^k}(\underline{\check{K}}_{G_{jm(l+3)v}}^{B^k,i}\Phi_{sv}^k), \tag{A59}$$

$$G_{slm}^{17,B_j^k} = \sum_{i=1}^{\nu^k}(\underline{\check{K}}_{G_{j(m+3)(l+3)v}}^{B^k,i}\Phi_{sv}^k), \tag{A60}$$

$$G_{qtl}^{18,B_j^k} = \sum_{i=1}^{\nu^k}(\underline{\hat{K}}_{G_{qlv}}^{B^k,i}\Phi_{tv}^k + \varepsilon_{lmn}\rho_m^{k,i}\underline{\hat{K}}_{G_{q(n+3)v}}^{B^k,i}\Phi_{tv}^k), \tag{A61}$$

$$G_{qrtl}^{19,B_j^k} = \sum_{i=1}^{\nu^k}(\varepsilon_{lmn}\phi_{rm}^{k,i}\underline{\hat{K}}_{G_{q(n+3)v}}^{B^k,i}\Phi_{tv}^k), \tag{A62}$$

$$G_{qtl}^{20,B_j^k} = \sum_{i=1}^{\nu^k}(\varepsilon_{lmn}\rho_m^{k,i}\underline{\hat{K}}_{G_{q(n+3)v}}^{B^k,i}\Phi_{tv}^k). \tag{A63}$$